THE PROBLEM OF ADJOINING ROOTS

TO ORDERED GROUPS

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DECLARATION

Except where explicitly stated, the work contained herein is my own.

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ABSTRACT

A long-standing open question in ordered group theory, first proposed by B.H. Neumann, is the following:-

Can an orderable group (O-group) be embedded in a divisible O-group?

This thesis, alas, contains no complete answer to this question. However, the sufficient conditions presented make something of a dint in the (to date) impregnable armour protecting this most vexing problem.

An *O-group* is a group, G, which can be linearly ordered in such a way that if $a \leq b$, then $ac \leq bc$ and $ca \leq cb$ (for a, b, c in G). A group, G, is *divisible* if for all g in G and integers $n > 0$, the equation $x^n = g$ has a (not necessarily unique) solution for x in G.

Chapter 1 contains an example of an O-group in which "long" commutators misbehave (in a certain sense). It is known (see B.H. Neumann [21]) that for elements a and b of an O-group, G, and integers $m \neq 0$:-

$$[a^m, b] = 1 \text{ implies } [a, b] = 1 ,$$

and

$$[a^m, b, a] = 1 \text{ implies } [a, b, a] = 1 .$$

Given an arbitrary, fixed subset T of the (strictly) positive integers, we construct an O-group, G, possessing elements g and h such that

$$\begin{aligned}[g^m, h, g, g, \ldots, g] &\neq 1 \text{ if } m \in T , \\ &= 1 \text{ if } m \notin T ,\end{aligned}$$

where the commutator has weight greater than, or equal to, 4.

The embedding theorem (Theorem 3) of Chapter 2 generalizes theorems of Kopytov [16] and Conrad [3]. Theorem 3 says, in effect, that we can suitably adjoin roots to an element, a, of an O-group, G, if a lies in a normal, abelian subgroup of G. That such a weak condition is (apparently) the strongest known sufficient condition for adjoining roots to an element of an O-group, admirably demonstrates the difficulty of B.H.

Neumann's question.

In Chapters 3 and 4 we turn our attention to the metabelian case. Even this case is proving obdurate, although, at the time of writing, is showing signs of weakening. Let G be a (non-abelian) metabelian O-group with A the isolated closure of the derived group of G. Then A is a normal, abelian proper subgroup of G, and the best we can say at this stage is that G can be embedded in a divisible, metabelian O-group if G/A is divisible. An example in Chapter 4 shows that if G always has a divisible, metabelian, orderable extension, then a minimal such extension need not be unique (up to isomorphism).

CHAPTER 0

INTRODUCTION AND PRELIMINARIES

0.0 Introduction

The problem of adjoining roots to (fully) ordered groups has been with us for some time. First considered by B.H. Neumann a quarter of a century ago, and since tackled by numerous mathematicians, it has been (and still is) the most vexing problem in the theory of ordered groups. As with most difficult mathematical problems, its statement is straightforward, but to understand the *complexity* of the problem requires a certain amount of "grass roots" contact (and frustration) with the problem. It is the fond hope of the author that the ensuing pages will help to revive interest in this difficult area of ordered group theory.

(In this section, we shall give neither definitions nor explanations of notation. This course is adopted for the sake of the continuity of the narrative. The reader unfamiliar with any terms used is referred to the next section (0.1 Definitions and notation).)

B.H. Neumann's question, which we quote almost directly from Fuchs ([7], p. 211), is the following:-

Let G *be an orderable group (O-group),* a *an element of* G *and* $n > 0$ *an integer.*

(I) *When can* G *be embedded in an* O*-group,* H, *containing* x *satisfying* $x^n = a$?

(II) *In which cases can a prescribed full order of* G *be extended to a full order of* H ?

One class of O-groups for which part (I) of this question has been answered is the class of locally nilpotent O-groups. Mal'cev, in [19] and [18] respectively, has shown that every locally nilpotent, torsion-free group is orderable and that every locally nilpotent, torsion-free group can be embedded in a divisible, locally nilpotent, torsion-free group. To the best of my knowledge, the answer to (II) in the case of locally nilpotent O-groups is not known (or if known, then certainly unsung). I suspect that the answer in this case is "always" and, as a step in the right direction, Theorem 5 of Chapter 2 gives such an answer when G is nilpotent of class 2.

Conrad [3] answers part (I) of the question for another class of O-groups. His Remark, p. 525, shows that if $(G, \leq)$ is an o-group of rank 2 (that is, G has precisely one proper convex subgroup), then G can be embedded in a divisible o-group, $(H, \leq')$, of rank 2 where $\leq'$ extends $\leq$.

In the same paper, Conrad gives a sufficient condition for n-th roots to be suitably adjoined to an element, a, of an o-group, G. The condition (implicit in his Theorem 3.1) is that a lies in an abelian, convex subgroup of G. Kopytov [16] gives another sufficient condition; namely, that a is in the centre of G. In Chapter 2 of this thesis, we generalize these results by showing (2.2 Corollary 1) that if a lies in a normal, abelian subgroup of an O-group, G, and $n > 0$ is an integer, then there is an O-group, H, containing G, and an element, x, in H such that $x^n = a$. Furthermore, any order of G extends to an order of H. Also, Theorem 3 of Chapter 2 answers the second half of a question of Kokorin ([14], question 1.61. He asks whether an O-group can be embedded in an O-group with divisible, maximal, locally nilpotent, normal subgroup (or divisible, maximal, abelian, normal subgroup)). The method used in Section 2.1 of Chapter 2 is almost entirely, that used by Kopytov (*op. cit.*).

One problem faced by the would-be adjoiner of roots to O-groups is that if an n-th root of a is to be adjoined to an O-group, G, such that the resulting group, H, is to be an O-group, then this root (call it x) must be well-behaved in certain respects. For instance, if f, g are in G, then $[x, g] = 1$ in H if, and only if, $[a, g] = 1$ in G, and $[x, g, x] = 1$ in H if, and only if, $[a, g, a] = 1$ in G. However, the behaviour of x is not governed completely by that of a. If $[a, f, g] = 1$ in G (where $a^r \neq g^s$ for all r, s in $\mathbb{Z}\backslash\{0\}$), we need not insist that $[x, f, g] = 1$ in H. (For these results of B.H. Neumann [21], see Fuchs [7], p. 38.) Furthermore, if e, f, g are in G and if $[a, e, f, g] = 1$ or $[e, f, a, g] = 1$, then we need not insist, respectively, that $[x, e, f, g] = 1$ or $[e, f, x, g] = 1$ in H. (See Chehata and Shawky [2].) In fact, take

$$g = [g_1, g_2, \ldots, g_{l-1}, a, g_{l+1}, \ldots, g_k]$$

in G, where l and k are positive integers satisfying

$$\left.\begin{array}{l} k > 3 \text{ and } k > l, \\ \textit{or} \\ k = 3, \; k > l \text{ and } a^r \neq g_k^s \text{ for all } r, s \text{ in } \mathbb{Z}\backslash\{0\}. \end{array}\right\} \quad (0.0.1)$$

If $g = 1$ or $\neq 1$, then we need not insist that $[g_1, g_2, \ldots, g_{l-1}, x, g_{l+1}, \ldots, g_k] = 1$ or $\neq 1$ respectively in H (see Chapter 1, Theorem 1).

This knowledge may make the problem easier to tackle. Our would-be adjoiner can say, "I'll take a new symbol, x, let H be generated by G and x, and set $x^n = a$. Also, I insist that, for g in G, $[x, g] = 1$ in H if, and only if, $[a, g] = 1$ in G, and that $[x, g, x] = 1$ in H if, and only if, $[a, g, a] = 1$ in G. For longer commutators $[g_1, g_2, \ldots, g_{l-1}, a, g_{l+1}, \ldots, g_k]$ in G, with l, k satisfying conditions (0.0.1), I shall simply let $[g_1, g_2, \ldots, g_{l-1}, x, g_{l+1}, \ldots, g_k] = 1$ and see what happens". This sort of approach seems to deserve exploration.

The third and fourth chapters of this thesis are concerned with another question of Kokorin ([14], question 1.60). He asks whether any metabelian, O-group can be embedded in a divisible, (metabelian) O-group. In Chapter 3, we outline a method for embedding a metabelian O-group in a divisible metabelian O-group. Of the four steps required, only one remains to be solved and Chapter 4 is devoted to a discussion of this unsolved step. The main result of Chapter 3 is 3.3 Theorem 4, which says that if G is a metabelian O-group and G/A is divisible (where A is the isolated closure of the derived group of G), then G can be embedded in a divisible, metabelian O-group, H . (This theorem is not as obvious as it may appear at first glance. See Section 3.3.)

The stumbling block preventing a complete solution, at this stage, to Kokorin's question is the following proposition:

$\underline{P}$ Let A be a divisible, abelian O-group with Φ an O-automorphism group of A . Then A can be extended to a divisible, abelian O-group, $\hat{A}$, such that $\Phi^{\#}$ (the minimal, divisible, abelian extension of Φ) is an O-automorphism group of $\hat{A}$.

The outcome of Kokorin's question will be determined by the truth, or otherwise, of this proposition (see Chapter 4, Theorem 1). If true, then the answer is "yes"; if false, then the answer is "not always". I shall make so bold as to conjecture that $\underline{P}$ is true.

The most interesting result of Chapter 4 is provided by an example (Section 4.4). This shows that if $\underline{P}$ is true, then a minimal, divisible, metabelian, orderable extension of a given metabelian O-group need not be unique (up to isomorphism).

Appendix 1 contains the presentation of a group, E_0 , which has the group E (see Chapter 1) as a homomorphic image. This presentation of E_0 may be of assistance to the reader when tackling Chapter 1. Appendix 2 is merely a reprint of my paper [5] published in 1972. The result contained therein, that the group with presentation $\langle a, c : a^{-1}c^2a = c^2a^2c^2 \rangle$ is not an O-group, was known (but unpublished) well before [5] appeared. Charles Holland ([10], 1960!) announced the existence of an "R-group which is not an O-group". Late in 1973, A.M.W. Glass informed me that Holland's example was the group mentioned above. So, the result which was to have occupied the opening pages of this thesis, and which was to have provided the justification for continued and, perhaps, increased interest in B.H. Neumann's question, has been relegated to the relative obscurity of the appendix.

This is not to deny the importance of the result or the hope that interest in the question will increase. Having "tinkered" with the problem for some time, both in the general case and in special cases, my feeling is that the question is not only answerable but affirmatively so. In view of the commutator results in Chapter 1 (and especially if the metabelian case works as planned) I shall (and, I hope, others will) approach the general problem with considerable optimism.

0.1 Definitions and notation

Throughout this thesis, identities in multiplicative groups will be denoted by 1, zeros in additive groups by 0, and (except when extra clarity is required) all orders by $\leq$. Basic set and group theoretic concepts will be assumed known to the reader. We shall use the following notation in the sequel, generally without further comment.

For sets $X, Y, X_j, X_{j+1}, \ldots, X_k$ ($j \leq k$ are integers), we have:-

$x \in X$	x is an element of X
$\{x \in X : \ldots\}$	the set of all $x \in X$ with property ...
$X \subseteq Y$	X is a subset of Y
$X \cap Y$, $X \cup Y$	the intersection and union, respectively, of X and Y

$X \backslash Y$	the set of all elements which are in X but not in Y
$X \times Y$	the cartesian product of X and Y
$\prod_{i=j}^{k} X_i$	cartesian product of $X_j, X_{j+1}, \ldots, X_k$

Let G and H be (multiplicative) groups, let $g, g_1, g_2, \ldots, g_k$ be elements of G, let X be a subset of G, and let $\{G_i : i \in I\}$ be a family of groups.

$G \leq H$	G is a subgroup of H
G'	the derived group of G
$\mathrm{gp}(X)$	the subgroup of G generated by the subset, X, of G. (If the elements of X can be listed $x_1, x_2, \ldots$, *or* if $X = \{g \in G : \ldots\}$, *or* if $X = Y \cup Z$ (where $Y \subseteq X$ and $Z \subseteq X$), then we write $\mathrm{gp}(x_1, x_2, \ldots)$, *or* $\mathrm{gp}\{g \in G : \ldots\}$, *or* $\mathrm{gp}(Y, Z)$, respectively, for $\mathrm{gp}(X)$.)
$I(X)$	the isolated closure of X in G (see definitions below)
H/G	the factor group of H with respect to its normal subgroup, G
$\mathrm{aut}(G)$	the group (under composition) of all automorphisms of G
$\prod_{j=1}^{k} g_j$	the product $g_1 g_2 \cdots g_k$
$[g_1, g_2, \ldots, g_k]$	defined inductively by $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$ and $[g_1, g_2, \ldots, g_k] = [[g_1, g_2, \ldots, g_{k-1}], g_k]$
$[g_1, ng]$	defined inductively by $[g_1, 1g] = [g_1, g]$ and $[g_1, ng] = [[g_1, (n-1)g], g]$

$\prod_{i \in I} G_i$ the *restricted* direct product of the family $\{G_i : i \in I\}$. (If no confusion arises, we may write $\prod_i G_i$.)

$\prod_{i \in I} g_i$ the standard form of an element in $\prod_{i \in I} G_i$. (Each g_i is in G_i and *at most finitely many* g_i are not equal to 1.)

We re-emphasize that *all* direct products considered in the sequel will be *restricted.*

The following, additional group theoretic concepts are required.

G is an *R-group* if, for all g in G and integers $n > 0$, the equation $x^n = g$ has at most one solution in G.

g in G is *generalized periodic* if

$$\left(g_1^{-1} g g_1\right)\left(g_2^{-1} g g_2\right) \cdots \left(g_k^{-1} g g_k\right) = 1 .$$

G is an *NGP-group* (notation of Hollister [12]) if the only generalized periodic element in G is 1.

If G is a subgroup of H, then G is *isolated* (*in* H) if, for h in H and integer $m > 0$, $h^m \in G$ implies $h \in G$. The *isolated closure of* G (*in* H) is the smallest, isolated subgroup of H containing G and will be denoted by $I(G)$.

We turn now to definitions and notation in ordered groups, most of which are to be found in Fuchs' Partially Ordered Algebraic Systems [7].

Let G and H be (multiplicative) groups.

po-GROUPS

The most frequently used order relation (that is, a binary relation which is reflexive, antisymmetric and transitive) is a partial order, and although this thesis is concerned almost exclusively with fully ordered groups, we define, first, partially ordered group. (Hereafter we say "relation" in place of "binary relation".)

The ordered pair, $(G, \leq)$ is a *partially ordered group* (denoted by *po-group*) if $\leq$ is a relation which makes $(G, \leq)$ a partially ordered set and which satisfies:-

for a, b, c, d in G, $a \leq b$ implies $cad \leq cbd$.

As usual, we write, for a, b in $(G, \leq)$, $a \geq b$ if $b \leq a$, and $a < b$ $(a > b)$ if $a \leq b$ $(a \geq b)$ and $a \neq b$.

The *positive cone of* $(G, \leq)$ is the set $\{g \in G : g > 1\}$ and is denoted by G^+. (Sometimes we use P to denote the positive cone of a partial order of G. In this case, we write P *is a partial order of* G.)

Z, Q, R, Z^+, Q^+, R^+

Z, Q and R denote, respectively, the additive groups of integers, rational numbers and real numbers. They are each endowed with their usual order with positive cones Z^+, Q^+ and R^+.

THE PREFIXES *o*- AND *O*-

These can be prefixed to either algebraic structures or mappings.

A *fully ordered group* (*o-group*) is a *po*-group, $(G, \leq)$, with the additional condition that the order relation is a linear order. That is, for all a, b in $(G, \leq)$, either $a \leq b$ or $b \leq a$. G is an *orderable group* (*O-group*) if, under some relation, $\leq$, G is an *o*-group. An order relation which makes G an *o*-group is called an *order of* G. (Sometimes, for the sake of clarity, we call such a relation a *full order of* G.)

When *o*- is prefixed to a mapping it means *order preserving*. So, if $(G, \leq)$ and $(H, \leq')$ are *po*-groups, then a homomorphism, $\phi : G \to H$, is an *o-homomorphism* if $a \leq b$ in G implies $a\phi \leq b\phi$ in H. *o-isomorphism* is defined similarly. (Since we shall be dealing with

o-isomorphisms between *fully* ordered groups, there will be no need to distinguish between an o-homomorphism *onto* and an o-epimorphism as does Fuchs [7], p. 21.) If $H = G$, and $\leq$ and $\leq'$ are *identical* (that is, they have the same positive cone), then we can define, similarly, *o-endomorphism* and *o-automorphism*. The set of all o-automorphisms of a po-group, $(G, \leq)$, forms a group under composition. This group we call the *o-automorphism group of* $(G, \leq)$ and denote it by $o\text{-aut}(G, \leq)$. If no confusion arises, we write, simply, *o-automorphism group of* G and $o\text{-aut}(G)$.

We apply the prefix O- to automorphism (although it could be applied easily to the other morphisms). An automorphism, ϕ, of an O-group, G, is an *O-automorphism* if, for some order of G, ϕ is an o-automorphism. Hence, a subgroup, Φ, of $\text{aut}(G)$ (where G is an O-group) is called an *O-automorphism group of* G, if, for some order of G, Φ is a subgroup of $o\text{-aut}(G)$.

INDUCED ORDER OF A SUBGROUP

Let G be a subgroup of the po-group $(H, \leq)$. For a, b in G, define $a \leq' b$ iff $a \leq b$ in H. (This is equivalent to defining $G^+ = H^+ \cap G$.) It is seen very easily that $(G, \leq')$ is a po-group, and the order, $\leq'$, is called the *order of* G *induced by the order*, $\leq$, *of* H. It follows immediately that any subgroup of an O-group is itself an O-group.

EXTENDING ORDERS

Let G be a subgroup of H, and let P be a partial order of G. We can *extend* P *to a partial order of* H if Q is a partial order of H and $P \subseteq Q$.

THE PREFIXES o^*- AND O^*-

G is an *O^*-group* if every partial order of G extends to a full order of G.

An embedding of G in an O-group, H, is an *o*-embedding* if every full order of G extends to a full order of H.

THE PREFIXES d- AND d^*-

G is *divisible* if for all g in G and n in $\mathbb{Z}^+$, the equation $x^n = g$ has a solution in G. If H is an extension of G, then H is a *d-extension* if H is a divisible group. H is a *d-closure* of G if H is a minimal d-extension of G. (This is Conrad's notation [3].) If H is an O-group and a d-extension of G, and if, in addition, any order of G extends to an order of H, then H is a *d*-extension* of G. H is a *d*-closure* of G if H is a minimal d^*-extension of G.

CONVEX SUBGROUPS

Let $(G, \leq)$ be a po-group and let C be a subgroup of G. Then C is *convex (in* G*)* if $1 < g < c$ implies g is in C, for g in G and c in C.

0.2 Preliminary results

On several occasions in the sequel we shall find ourselves in the position of having a group, G, and a relation, $\leq$, on G, and wondering whether or not $(G, \leq)$ is an o-group. On these occasions, we shall apply the following theorem, which is, virtually, a restatement of Theorem 2 plus Proposition 3 (c) of Fuchs [7], p. 13, and will not be proved here. (Note that our positive cones do not contain the identity while those of Fuchs do.)

THEOREM 1. *Let* G *be a group and let* $\leq$ *be a (binary) relation on* G. *Then* $(G, \leq)$ *is an* o*-group if, and only if, the following are satisfied for* g, g' *in* G :-

(i) $g > 1$ *implies* $g^{-1} \not> 1$,

(ii) $g \in G\backslash\{1\}$ *implies* $g > 1$ *or* $g^{-1} > 1$,

(iii) $g > 1$ *and* $g' > 1$ *imply* $gg' > 1$, *and*

(iv) $g' > 1$ *implies* $g^{-1}g'g > 1$.

The positive cone, G^+ , of a *po*-group, $(G, \leq)$ has the following useful property.

THEOREM 2 (Fuchs [7], p. 15, Proposition 5). *No element in* G^+ *is generalized periodic.*

The following obvious corollary is well known.

COROLLARY 1. *Every O-group is an NGP-group.*

We shall use the following (slightly) more convenient characterization of *o*-homomorphism than that given in 0.1.

THEOREM 3. *Let* $(G, \leq)$ *and* $(H, \leq)$ *be po-groups (where the orders need not be identical) and let* $\phi : G \to H$ *be a homomorphism. Then* ϕ *is an o-homomorphism if, and only if,* $g \geq 1$ *in* G *implies* $g\phi \geq 1$ *in* H .

Proof. See Fuchs [7], p. 20. //

(Note that when we apply this theorem to an automorphism, χ , of G , we have:- χ is an *o*-automorphism of G if, and only if, $g > 1$ implies $g\chi > 1$.)

The next theorem is a special case of a Theorem in Fuchs (Abelian Groups) [6].

THEOREM 4. *Let* A *and* B *be abelian groups such that* B *is torsion-free and* $\phi : A \to B$ *is a homomorphism. Let* $A^{\#}$ *and* $B^{\#}$ *be abelian d-closures of* A *and* B *respectively. Then* ϕ *can be extended uniquely to a homomorphism* $\phi^{\#} : A^{\#} \to B^{\#}$.

Proof. The existence of $\phi^{\#}$ is given by Theorem 16.1 [6]. The uniqueness follows from two subsequent remarks in [6]. (Namely, the last sentence before §17, p. 60 and the last paragraph of p. 66.) //

In fact, the homomorphism, $\phi^{\#}$, is given by

$$a\phi^{\#} = (a^m\phi)^{1/m} \quad \text{where } a \in A^{\#} , \ a^m \in A \text{ and } m \in Z^+ \tag{0.2.1}$$

(*cf.* Conrad [3], Lemma, p. 518). We emphasize (and Conrad shows) that the definition of $\phi^{\#}$ is *not* dependent on the choice of m in Z^+ such that a^m is in A.

0.3 o-semi-direct products

Semi-direct products (see, for example, Kurosch [17], p. 149) play an important part in the work to follow and, hence, deserve a section of their own. The example in Chapter 1 is built up from a basic group (itself a semi-direct product) using only semi-direct products, and Theorem 6 (of this section) allows us, in Chapters 3 and 4, to handle split extensions with greater ease.

We define a semi-direct product of groups G and H as follows:-

Suppose there is an *action* of H on G. That is, for all g in G and h in H, there is an element, g^h, in G such that the mapping $g \mapsto g^h$ (h fixed, g ranging over G) is an automorphism of G, and such that for all g in G and h, h' in H,

$$g^{(hh')} = (g^h)^{h'} . \tag{0.3.1}$$

The *semi-direct product of* G *by* H (denoted by $G\lambda H$) is the set $H \times G$ with multiplication given by

$$(h, g)(h', g') = (hh', g^{h'}g') \tag{0.3.2}$$

for all h, h' in H and g, g' in G. The mappings of G and H into $G\lambda H$ given, respectively, by

$$g \mapsto (1, g) \quad \text{for all } g \text{ in } G \tag{0.3.3}$$

and

$$h \mapsto (h, 1) \quad \text{for all } h \text{ in } H \tag{0.3.4}$$

are embeddings. It is often convenient to use the following characterization of semi-direct products.

THEOREM 5 (see Kurosch *op. cit.*). *Let* F *be a group with subgroups* G *and* H. *Then* F *is a semi-direct product of* G *by* H *if, and only if,*

(i) G *is normal in* F,

(ii) $G \cap H = \{1\}$, *and*

(iii) $F = \mathrm{gp}(G, H)$.

Now, (0.3.1) implies that, for all h in H, the mapping which takes h to the automorphism $g \mapsto g^h$ is a homomorphism from H into $\mathrm{aut}(G)$. In fact, distinct homomorphisms from H into $\mathrm{aut}(G)$ give rise to distinct semi-direct products of G by H (Kurosch *op. cit.*). (Whenever, in the sequel, we have a semi-direct product, $G\lambda H$, we shall be careful to leave no doubt as to which action of H on G defines $G\lambda H$.)

If G and H are O-groups, not all semi-direct products of G by H need be O-groups. (For example, the group with presentation $(a, b : a^{-1}ba = b^{-1})$ is a semi-direct product of two infinite cycles but is clearly not an O-group.) We modify the definition given above as follows:-

Let G and H be o-groups with an *o-action* of H on G. That is, there is an action of H on G satisfying

$$g > 1 \text{ implies } g^h > 1 \qquad (0.3.5)$$

for all $g > 1$ in G and h in H. The *o-semi-direct product of* G *by* H is $G\lambda H$ (formed using the given action of H on G) with order defined by

$$(h, g) > 1 \text{ iff } h > 1 \text{ in } H, \text{ or } h = 1 \text{ in } H \text{ and } g > 1 \text{ in } G. \qquad (0.3.6)$$

To justify the use of the prefix o- here, we have:-

LEMMA 1. *Let* G *and* H *be o-groups with an o-action of* H *on* G. *Then* $G\lambda H$ *is an o-group under the order given by* (0.3.6).

This result is known and we shall omit the proof (which is simply a straightforward verification of *(i)*, *(ii)*, *(iii)* and *(iv)* of 0.2 Theorem 1). Conrad [3] mentions orderable, split extensions (p. 518, II, and see the definition of order, p. 517, last paragraph).

We conclude this section with the theorem mentioned earlier.

THEOREM 6. *Let G and H be O-groups with an action of H on G. Let φ be the corresponding homomorphism from H into* $\mathrm{aut}(G)$. *Then $G\lambda H$ is an O-group if, and only if, $H\varphi$ is an O-automorphism group of G.*

Proof. Suppose $F = G\lambda H$ is an O-group and let $\leq$ be any order of F. For all g in G and h in H, $(h, 1)^{-1}(1, g)(h, 1) = (1, g^h)$. So, $(1, g) > 1$ in F implies $(1, g^h) > 1$ in F (by 0.2 Theorem 1 *(iv)*). This means that $g > 1$ in G implies $g^h > 1$ in G (where the order of G, a subgroup of F, is that induced by the chosen order of F). Since $h\varphi$ is the automorphism $g \mapsto g^h$, it follows that $H\varphi$ is an O-automorphism group of G.

Conversely, suppose that $H\varphi$ is an O-automorphism group of G. Since $g^h = g(h\varphi)$ for all g in G and h in H, an order of G can be chosen with respect to which the given action of H on G is an O-action. Choose, in addition, any order of H and order F by (0.3.6). //

CHAPTER 1

COMMUTATORS IN ORDERABLE GROUPS: AN EXAMPLE

1.0 Introduction

It is known (see B.H. Neumann [21]) that in any O-group, G, the following implications hold for g and h in G and m in $\mathbb{Z}\setminus\{0\}$:-

$$[g^m, h] = 1 \text{ implies } [g, h] = 1 \tag{1.0.1}$$

and

$$[g^m, h, g] = 1 \text{ implies } [g, h, g] = 1 . \tag{1.0.2}$$

(In fact, (1.0.1) holds in any R-group (see Kurosh [17], p. 244) and it can be shown that (1.0.2) is a consequence of (1.0.1). Furthermore, a group, H, is an R-group if, and only if, H is torsion-free and has property (1.0.1).)

In answer to a question of B.H. Neumann (implicit in [21]), we give an example of an O-group possessing elements, g and h, such that for all m in $\mathbb{Z}^+$ and $n \geq 2$ in $\mathbb{Z}^+$,

$$\begin{aligned}[g^m, h, ng] &\neq 1 \text{ if } m \in T \\ &= 1 \text{ if } m \notin T ,\end{aligned}$$

where T is an arbitrary, fixed subset of $\mathbb{Z}^+$.

In Section 1.1 we describe a method of constructing o-groups, and in Section 1.2 we use this construction twice, then adjoin an o-automorphism, to obtain the example mentioned above. In Section 1.3, we show that in any metabelian O-group, the following holds for all $g_1, g_2, \ldots, g_n$ in the group $(n \geq 1)$ and m in $\mathbb{Z}\setminus\{0\}$:-

$$\left[g_1, g_2, \ldots, g_{l-1}, g_l^m, g_{l+1}, \ldots, g_n\right] = 1 \text{ if, and only if,}$$

$$[g_1, g_2, \ldots, g_{l-1}, g_l, g_{l+1}, \ldots, g_n] = 1 . \tag{1.0.3}$$

This result, which may be known, is presented here both for use in Chapter

4 and because it is of some interest in itself.

(Throughout this chapter, the following commutator identities will be used with, or without, comment:-

$$[x, y]^{-1} = [y, x]$$

$$[x, y] = [y, x^{-1}]^{x} = [x^{-1}, y]^{-x}$$

$$[xy, z] = [x, z]^{y}[y, z]$$

$$[x, yz] = [x, z][x, y]^{z}$$

$$[x, y]^{z} = [x^{z}, y^{z}] \text{ .)}$$

1.1 An o-group construction

Let $\{A_i : i \in \mathbb{Z}\}$ be a family of o-groups. (Throughout this section, $a(i)$, $x(i)$, $y(i)$ denote arbitrary elements of A_i for each i in $\mathbb{Z}$.) Suppose the following hold for all $i < j < k$ in $\mathbb{Z}$:-

$$\text{there is an } o\text{-action of } A_j \text{ on } A_i \text{ ,} \tag{1.1.1}$$

$$a(i)^{(a(j)^{a(k)})} = a(i)^{a(j)} \text{ ,} \tag{1.1.2}$$

$$(a(i)^{a(j)})^{a(k)} = (a(i)^{a(k)})^{a(j)} \text{ .} \tag{1.1.3}$$

(Note that once the A_i are embedded in a group with the given actions corresponding to conjugation in the group (and this is our aim), then (1.1.2) will be a consequence of (1.1.3).)

We define, inductively, a sequence of groups, $B_0, B_1, B_2, \ldots$, where elements of $B_{2(k-1)}$ and B_{2k-1} (for k in $\mathbb{Z}^+$) are of the form $(a(k-1), a(k-2), \ldots, a(-k+1))$ and $(a(k), a(k-1), \ldots, a(-k+1))$ respectively.

Define $B_0 = A_0$ and, for k in $\mathbb{Z}^+$, $B_{2k-1} = B_{2(k-1)}\lambda A_k$ and $B_{2k} = A_{-k}\lambda B_{2k-1}$, where each of these is an o-semi-direct product with

o-actions given, respectively, by

$$\big(a(k-1),\ a(k-2),\ \ldots,\ a(-k+1)\big)^{a(k)}$$
$$= \big(a(k-1)^{a(k)},\ a(k-2)^{a(k)},\ \ldots,\ a(-k+1)^{a(k)}\big) \quad (1.1.4)$$

and

$$a(-k)^{\big(a(k),a(k-1),\ldots,a(-k+1)\big)} = \Big(\ldots\Big(\big(a(-k)^{a(k)}\big)^{a(k-1)}\Big)\ldots\Big)^{a(-k+1)} . \quad (1.1.5)$$

(1.1.1), (1.1.2) and (1.1.3) ensure that (1.1.4) and (1.1.5) are o-actions. Hence, B_{2k-1} is the set, $\prod_{i=k}^{-k+1} A_i$, with multiplication

$$\big(x(k),\ x(k-1),\ \ldots,\ x(-k+1)\big)\ \big(y(k),\ y(k-1),\ \ldots,\ y(-k+1)\big)$$
$$= \big(x(k)y(k),\ x(k-1)^{y(k)}y(k-1),\ \ldots$$
$$\ldots,\ \Big(\ldots\Big(\big(x(-k+1)^{y(k)}\big)^{y(k-1)}\Big)\ldots\Big)^{y(-k+2)}y(-k+1)\big)$$

and order $\big(x(k),\ x(k-1),\ \ldots,\ x(-k+1)\big) > 1$ if, and only if,

$$x(k) = 1,\ x(k-1) = 1,\ \ldots,\ x(k-j+1) = 1 \ \text{ and } \ x(k-j) > 1$$

for some $j = 1, 2, \ldots, 2k-1$.

B_{2k} is the set, $\prod_{i=k}^{-k} A_i$, with similar multiplication and order.

Now, there is a natural embedding of each B_i into B_{i+1} ($i \geq 0$ in $\mathbb{Z}$). Namely, a mapping of the form (0.3.3) if i is even and one of the form (0.3.4) if i is odd. So, we let C be the direct limit of the groups $B_0, B_1, B_2, \ldots$. Elements of C will be denoted by (formal) products, $\prod_{s=1}^{u} a\big(i(s)\big)$, where $i(1) > i(2) > \ldots > i(u)$ in $\mathbb{Z}$. By the definition of C , for any two elements, x and y , in C , there is u in $\mathbb{Z}^+$ such that $x = \prod_{s=1}^{u} x\big(i(s)\big)$ and $y = \prod_{s=1}^{u} y\big(i(s)\big)$. So,

$$xy = \prod_{s=1}^{u} x\big(i(s)\big)^{y(i(1))y(i(2))\ldots y(i(s-1))}\, y\big(i(s)\big) .$$

The order given by

$x > 1$ iff $x(i(1)) = 1$, $x(i(2)) = 1$, ..., $x(i(j-1)) = 1$ and $x(i(j)) > 1$

for some $j = 1, 2, \ldots, u$, makes C an o-group. We shall denote this o-group by $\bigwedge_{i \in Z} A_i$.

In some cases there is a natural o-automorphism of C.

LEMMA 1. *Suppose that for all i in Z, ι_i is an o-isomorphism from A_i onto A_{i+1} such that for all $i < j$ in Z,*

$$\left(a(i)^{a(j)}\right)\iota_i = \left(a(i)\iota_i\right)^{\left(a(j)\iota_j\right)} . \tag{1.1.6}$$

Then the mapping, $\phi : C \to C$, given by

$$\left(\prod_{s=1}^{u} a\left(i(s)\right)\right)\phi = \prod_{s=1}^{u} a\left(i(s)\right)\iota_{i(s)} \tag{1.1.7}$$

is an o-automorphism of C.

Proof. (1.1.6) ensures that ϕ is an endomorphism of C. ϕ is one-to-one, onto and order preserving because each ι_i is an o-isomorphism from A_i onto A_{i+1}. //

1.2 The example

As mentioned in the introduction to this chapter, we shall use the construction of 1.1 twice, plus the process of adjoining an o-automorphism, to achieve the required result.

1.2.1. Let $\{A_i : i \in Z\}$ be the family of o-groups where each A_i is (isomorphic to) the group with presentation

$$\left(x, y ; y^{-1}xy = x^2\right) . \tag{1.2.1}$$

Elements of A_i will be written $a(i, m, \alpha)$, where m is in Z and $\alpha = r2^s$ for some r, s in Z (see Fuchs and Sasiada [9], p. 15). So, the defining relation in the presentation (1.2.1) becomes, for each i in Z,

$$a(i, 1, 0)^{-1}a(i, 0, 1)a(i, 1, 0) = a(i, 0, 2) \ . \qquad (1.2.2)$$

Multiplication in A_i is given by

$$a(i, m, \alpha)a(i, n, \beta) = a\left(i, m+n, 2^n\alpha+\beta\right) \ . \qquad (1.2.3)$$

Each A_i will be ordered by

$$a(i, m, \alpha) > 1 \text{ iff } m > 0 \text{ , or } m = 0 \text{ and } \alpha > 0 \ .$$

Let T be an arbitrary subset of Z^+. Define $\nu : Z^+ \times Z \to Z$ by

$$\begin{aligned} \nu(k, m) &= 1 - 2^m && \text{if } k \in T \\ &= 0 && \text{if } k \notin T \ . \end{aligned} \qquad (1.2.4)$$

For all $i < j$ in Z, we define an o-action of A_j on A_i by

$$a(i, m, \alpha)^{a(j,n,\beta)} = a\left(i, m, \nu(j-i, m)n+\alpha\right) \ . \qquad (1.2.5)$$

That (1.2.5) does define an o-action of A_j on A_i is shown easily using (1.2.3) and the identity $\nu(k, r+s) = 2^s\nu(k, r) + \nu(k, s)$ for all k in Z^+ and r, s in Z. Furthermore, (1.2.5) satisfies (1.1.2) and (1.1.3). Let $C = \bigwedge_{i \in Z} A_i$.

LEMMA 2. *In C, the following hold:-*

(i) for all $i < j$ in Z and n in Z^+,

$$\begin{aligned} \left[a(j, 1, 0)^{-1}, na(i, 1, 0)\right] &= a(i, 0, 1)^{-1} && \text{if } j - i \in T \\ &= 1 && \text{if } j - i \notin T \ ; \end{aligned} \qquad (1.2.6)$$

(ii) the mapping, $\phi : C \to C$, given by

$$\left(\prod_{s=1}^{u} a\left(i(s), m(s), \alpha(s)\right)\right)\phi = \prod_{s=1}^{u} a\left(i(s)+1, m(s), \alpha(s)\right) \qquad (1.2.7)$$

is an o-automorphism of C.

Proof. *(i)* To prove (1.2.6), it suffices to prove cases $n = 1$ and $n = 2$. Take $i < j$ in Z.

$$
\begin{aligned}
[a(j,1,0)^{-1}, a(i,1,0)] &= a(i,-1,0)^{a(j,-1,0)}a(i,1,0) \\
&= a(i,-1,-\nu(j-i,-1))a(i,1,0) && \text{(by (1.2.5))} \\
&= a(i,0,-\nu(j-i,-1)2) && \text{(by (1.2.3))} \\
&= \begin{cases} a(i,0,-2^{-1}2) & \text{if } j-i \in T \\ a(i,0,0) & \text{if } j-i \notin T \end{cases} && \text{(by (1.2.4))} \\
&= \begin{cases} a(i,0,1)^{-1} & \text{if } j-i \in T \\ 1 & \text{if } j-i \notin T . \end{cases}
\end{aligned}
$$

So,

$$
\begin{aligned}
[a(j,1,0), 2a(i,1,0)] &= [a(i,0,1)^{-1}, a(i,1,0)] && \text{if } j-i \in T \\
&= 1 && \text{if } j-i \notin T ,
\end{aligned}
$$

where

$$
\begin{aligned}
[a(i,0,1)^{-1}, a(i,1,0)] &= a(i,0,1)a(i,0,1)^{-2} \\
&= a(i,0,1)^{-1} .
\end{aligned}
$$

(ii) To prove (1.2.7), we take the o-isomorphism, ι_i , from A_i onto A_{i+1} given by $a(i, m, \alpha)\iota_i = a(i+1, m, \alpha)$ and show that (1.1.7) is satisfied. Take $i < j$ in Z .

$$
\begin{aligned}
\left(a(i,m,\alpha)^{a(j,n,\beta)}\right)\iota_i &= a(i,m,\nu(j-i,m)n+\alpha)\iota_i \\
&= a(i+1,m,\nu(j+1-(i+1),m)n+\alpha) \\
&= a(i+1,m,\alpha)^{a(j+1,n,\beta)} \\
&= \left(a(i,m,\alpha)\iota_i\right)^{(a(j,n,\beta))\iota_j} . \qquad //
\end{aligned}
$$

1.2.2. Let $\{C_r : r \in \mathsf{Z}\}$ be the family of o-groups where each C_r is (isomorphic to) the o-group, C , just constructed. In particular, for each r in Z , $C_r = \bigwedge_{i \in \mathsf{Z}} A_{r,i}$, where each $A_{r,i}$ has the presentation (1.2.1). Elements of $A_{r,i}$ will be written $a_r(i, m, \alpha)$. So, elements of C_r are of the form

$$c(r) = \prod_{s=1}^{u} a_r\big(i(s),\ m(s),\ \alpha(s)\big) \tag{1.2.8}$$

(see remarks preceding 1.1 Lemma 1). Let $\phi_r : C_r \to C_r$ be the o-automorphism given by (1.2.7) and define, for all r in Z, $\mu : \mathsf{Z}^+ \times C_r \to \mathsf{Z}$ by

$$\mu\big(k,\ c(r)\big) = \sum_{s=1}^{u} m(s) \quad \text{if } k = 1$$

$$= 0 \qquad\qquad \text{if } k \geq 2 \tag{1.2.9}$$

where $c(r)$ is of the form (1.2.8). (Although μ depends, in a sense, on r, no confusion will arise by not making any notational reference to this fact.) For all $r < r'$ in Z, define an o-action of $C_{r'}$ on C_r by

$$c(r)^{c(r')} = c(r)\phi_r^{w}\ , \text{ where } w = \mu\big(r'-r,\ c(r')\big)\ . \tag{1.2.10}$$

Since ϕ_r is an o-automorphism of C_r, and since the identity $\mu\big(k,\ c(r')c(r')'\big) = \mu\big(k,\ c(r')\big) + \mu\big(k,\ c(r')'\big)$ holds for all k in Z^+ and $c(r'),\ c(r')'$ in $C_{r'}$, (1.2.10) does define an o-action of $C_{r'}$ on C_r. Furthermore, (1.2.10) satisfies (1.1.2) (because $c(r)^{\left(c(r')\phi_{r'}\right)} = c(r)^{c(r')}$ for all $r < r'$ in Z), and (1.2.10) satisfies (1.1.3) (because $c(r)^{c(r'')} = c(r)$ for all $r < r' < r''$ in Z). Let $D = \bigwedge_{r\in\mathsf{Z}} C_r$, and let ι_r' be the o-isomorphism from C_r onto C_{r+1} given by

$$\left(\prod_{s=1}^{u} a_r\big(i(s),\ m(s),\ \alpha(s)\big)\right)\iota_r' = \prod_{s=1}^{u} a_{r+1}\big(i(s),\ m(s),\ \alpha(s)\big)\ .$$

LEMMA 3. *In D, the following hold:-*

(i) for all r, i, j in Z, and $n \geq 2$ and p in Z^+,

$$\left[a_{r+1}(i,\,1,\,0)^p,\, na_r(j,\,1,\,0)\right] = a_r(j,\,0,\,-2) \quad \textit{if} \quad p \in T$$
$$= 1 \qquad\qquad \textit{if} \quad p \notin T\,; \qquad (1.2.11)$$

(ii) the mapping, $\phi' : D \to D$ *, given by*

$$\left(\prod_{t=1}^{v} c(r(t))\right)\phi' = \prod_{t=1}^{v} c(r(t))\iota'_{r(t)} \qquad (1.2.12)$$

is an o-automorphism of D *.*

Proof. *(i)*

$$\left[a_{r+1}(i,\,1,\,0)^p,\, a_r(j,\,1,\,0)\right] = a_r(j,\,-1,\,0)^{a_{r+1}(i,p,0)}a_r(j,\,1,\,0)$$
$$= a_r(j+p,\,-1,\,0)a_r(j,\,1,\,0) \quad \text{(by (1.2.10))}\,.$$

So

$$\left[a_{r+1}(i,\,1,\,0)^p,\, na_r(j,\,1,\,0)\right]$$
$$= \left[a_r(j+p,\,-1,\,0),\,(n-1)a_r(j,\,1,\,0)\right]^{a_r(j,1,0)}$$
$$= \begin{cases} a_r(j,\,0,\,-1)^{a_r(j,1,0)} & \text{if } p \in T \\ 1 & \text{if } p \notin T \end{cases} \quad \text{(by (1.2.6))}$$
$$= \begin{cases} a_r(j,\,0,\,-2) & \text{if } p \in T \\ 1 & \text{if } p \notin T\,. \end{cases} \quad \text{(by (1.2.2))}$$

(ii) To prove (1.2.12), we show that each ι'_r satisfies (1.1.6). It is clear, from the definition of ι'_r, that $\mu(k,\,c(r)\iota'_r) = \mu(k,\,c(r))$ for all k in $\mathbb{Z}^+$ and r in $\mathbb{Z}$. Furthermore, it is clear that $(c(r)\phi_r)\iota'_r = (c(r)\iota'_r)\phi_{r+1}$ for all r in $\mathbb{Z}$. So, for all $r < r'$ in $\mathbb{Z}$, we have

$$\left(c(r)^{c(r')}\right)\iota'_r = \left(c(r)\phi^w_r\right)\iota'_r \qquad \text{(see (1.2.10))}$$

$$= \left(c(r)\iota'_r\right)\phi^w_{r+1}$$

$$= \left(c(r)\iota'_r\right)\phi^{w'}_{i+1}$$

$$\left(\text{where } w' = \mu\left(r'+1-(r+1),\ c(r')\iota'_{r'}\right) = \mu\left(r'-r,\ c(r')\right) = w\ \right)$$

$$= \left(c(r)\iota'_r\right)^{\left(c(r')\iota'_{r'}\right)} \qquad \text{(by (1.2.10))}. \qquad //$$

By way of a corollary to this lemma, we have:-

THEOREM 1. *In an O-group, G, with $g_1, g_2, \ldots, g_n$ in G $(n \geq 3)$ and $p \neq q$ in $\mathbb{Z}^+$, the following are compatible:-*

$$\left[g_1, g_2, \ldots, g_{l-1}, g_l^p, g_{l+1}, \ldots, g_n\right] \neq 1$$

and

$$\left[g_1, g_2, \ldots, g_{l-1}, g_l^q, g_{l+1}, \ldots, g_n\right] = 1 ,$$

where $l < n$.

Proof. First we stress that l must be strictly less than n, otherwise (1.0.1) shows that the situation described in the theorem cannot occur.

(1.2.11), in the case where T is neither empty nor the whole of $\mathbb{Z}^+$, proves the case $l = 1$. For $l > 1$, we have

$$\left[g_1, g_2, \ldots, g_{l-1}, g_l^m, g_{l+1}, \ldots, g_n\right] = \left[g_l^m, g, g_{l+1}, \ldots, g_n\right]^{-h}$$

for all m in $\mathbb{Z}$, where $g = \left[g_1, g_2, \ldots, g_{l-1}\right]$ and $h = \left[g, g_l^m\right]$. By (1.2.11), $\left[g_l^p, g, g_{l+1}, \ldots, g_n\right] \neq 1$ is compatible with $\left[g_l^q, g, g_{l+1}, \ldots, g_n\right] = 1$, and the result follows. //

1.2.3. Theorem 1 does not rule out the possibility that in any

O-group, G, $[g^m, h, ng] = 1$ implies $[g, h, ng] = 1$ for g, h in G and $n \geq 2$, $m \neq 0$ in $\mathbf{Z}$. To show that this implication is not true in general, we adjoin the o-automorphism, ϕ', to D. That is, E is the o-semi-direct product of D by an infinite cycle (with generator e), the o-action being induced by $d^e = d\phi'$ for all d in D.

LEMMA 4. *In E, the following holds:- for all r, i in $\mathbf{Z}$, and $n \geq 2$ and p in $\mathbf{Z}^+$,*

$$\left[a_r(i, 1, 0)^p, e, na_r(i, 1, 0)\right] \neq 1 \quad \text{if } p \in T$$
$$= 1 \quad \text{if } p \notin T. \qquad (1.2.13)$$

Proof. In any group, G, $[g^m, h, ng] = [h^{-1}g^m h, ng]$ holds for all g, h in G, m in $\mathbf{Z}$ and n in $\mathbf{Z}^+$. So,

$$\left[a_r(i, 1, 0)^p, e, na_r(i, 1, 0)\right] = \left[e^{-1}a_r(i, 1, 0)^p e, na_r(i, 1, 0)\right]$$
$$= \left[a_{r+1}(i, 1, 0)^p, na_r(i, 1, 0)\right]$$

and (1.2.13) follows from (1.2.11). //

As a corollary to this lemma, we have:-

THEOREM 2. *For each subset, T, of $\mathbf{Z}^+$, there is an O-group possessing elements, g and h, such that for $n \geq 2$ and m in $\mathbf{Z}^+$,*

$$[g^m, h, ng] \neq 1 \quad \text{if } m \in T$$
$$= 1 \quad \text{if } m \notin T.$$

Proof. Simply take E (of the previous lemma) to be the O-group and set $g = a_r(i, 1, 0)$ (for some r, i in $\mathbf{Z}$) and $h = e$. //

1.3. Commutators in metabelian O-groups

For this section, we adopt the following notation:-

$$g(n; l, m) = \left[g_1, g_2, \ldots, g_{l-1}, g_l^m, g_{l+1}, \ldots, g_n\right]$$

where $1 \leq l < n$ and m are in $\mathbf{Z}\backslash\{0\}$, and $g_1, g_2, \ldots, g_n$ are all elements of some (fixed) group. (We have made $l < n$ because (1.0.3) is an immediate consequence of (1.0.1) if $l = n$.)

Let G be a metabelian O-group. To prove (1.0.3), we start with the case $l = 1$.

LEMMA 5. *In* G *, the following holds:-*

$$g(n;\, 1,\, m) = \prod_{j=m-1}^{0} g(n;\, 1,\, 1)^{g_1^j} \qquad (1.3.1)$$

for all $1 < n$ *and* m *in* $\mathbb{Z}\backslash\{0\}$.

Proof. If $n = 2$, we apply the commutator identity $[x^m, y] = \prod_{j=m-1}^{0} [x, y]^{x^j}$. Given that (1.3.1) is true for all $2 \le n < p$ (p in $\mathbb{Z}^+$), we have

$$\begin{aligned} g(p;\, 1,\, m) &= [g(p-1;\, 1,\, m),\, g_p] \\ &= \left[\prod_{j=m-1}^{0} g(p-1;\, 1,\, 1)^{g_1^j},\, g_p\right] \\ &= \prod_{j=m-1}^{0} \left[g(p-1;\, 1,\, 1)^{g_1^j},\, g_p\right] \end{aligned}$$

(since G is metabelian and $g(p-1;\, 1,\, 1) \in G'$).

Now, for g, h in G and c in G' ,

$$\begin{aligned} [c^h, g] &= \left[c,\, g^{h^{-1}}\right]^h \\ &= [c,\, g[g,\, h^{-1}]]^h \\ &= [c,\, g]^h \;; \end{aligned}$$

whence

$$g(p;\, 1,\, m) = \prod_{j=m-1}^{0} [g(p-1;\, 1,\, 1),\, g_p]^{g_1^j}$$

where $[g(p-1;\, 1,\, 1),\, g_p] = g(p;\, 1,\, 1)$. //

LEMMA 6. *For* $1 < l < n$ *and* m *in* $\mathbb{Z}\backslash\{0\}$ *, the following holds in* G *:-*

$$g(n;\, l,\, m) = \prod_{j=m-1}^{0} g(n;\, l,\, 1)^{g_l^j} .$$

Proof. The identity $[x,\, y] = [x^{-1},\, y]^{-x}$ implies (modulo a straight-forward induction proof) that $g(n;\, l,\, m) = \left[g_l^m,\, h,\, g_{l+1},\, \ldots,\, g_n\right]^{-1}$, where $h = [g_1,\, g_2,\, \ldots,\, g_{l-1}]$. (This works because $l > 1$ implies that $\left[h,\, g_l^m\right]$ is in G' .) Applying (1.3.1) gives the required result. //

As a corollary to Lemma 6, we have:-

THEOREM 3. *In a metabelian* O*-group,* G , (1.0.3) *holds. That is, for* $g_1,\, g_2,\, \ldots,\, g_n$ *in* G *, and* $1 \le l \le n$ *and* m *in* $\mathbb{Z}\backslash\{0\}$ *,*

$$\left[g_1,\, g_2,\, \ldots,\, g_{l-1},\, g_l^m,\, g_{l+1},\, \ldots,\, g_n\right] = 1$$

if, and only if

$$[g_1,\, g_2,\, \ldots,\, g_{l-1},\, g_l,\, g_{l+1},\, \ldots,\, g_n] = 1 .$$

Proof. If $l = n$, use (1.0.1). If $l < n$, we observe that G is an NGP-group* (0.2 Corollary 1); whence,

$$\prod_{j=m-1}^{0} g(n;\, l,\, 1)^{g_1^j} = 1 \text{ iff } g(n;\, l,\, 1) = 1 . \qquad //$$

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* See Chapter 4, Section 4.5 - Addendum.

CHAPTER 2

AN EMBEDDING THEOREM

2.0 Introduction

In this chapter, a sufficient condition is given for adjoining roots to a given element of an O-group; namely, that the normal closure of the given element is abelian (Corollary 1). This is obtained by way of a result (Theorem 3) which generalizes theorems of Kopytov [16], Conrad [3] and Terehov [24], and which answers part of a question of Kokorin (see [14], question 1.61). In fact, the method used in Section 2.1 is almost identical to that employed by Kopytov (op. cit.). (His theorem appears also in Fuchs [8], p. 83.) Section 2.2 includes a result (Theorem 5) on the d-closure of an O-group which is nilpotent of class 2, and in Section 2.3, some further consequences of Theorem 3 are presented.

It is worth noting that the results of Section 2.1 are true if "O-group" is replaced throughout by "R-group" and if "o-automorphism" is replaced by "automorphism".

2.1 Completing a normal, abelian subgroup of an O-group

We begin with a definition. By *completing a subgroup,* U *, of a group,* V *,* we mean that V can be embedded in a group, W , in such a way that W contains a d-extension of (the image under the embedding of) U .

Suppose G is an O-group with normal, abelian subgroup, A . We wish to complete A and, for the moment, suppose that A is isolated.

For all g in G , denote by ϕ_g the restriction to A of the inner automorphism of G induced by g . (That is, $a\phi_g = g^{-1}ag$.) Let $A^{\#}$ be an abelian d-closure of A and for all a in $A^{\#}$, let $m(a)$ be a positive integer such that $a^{m(a)}$ is in A. Define $\phi_g^{\#} : A^{\#} \to A^{\#}$ by

$$a\phi_g^{\#} = \left(a^m\phi_g\right)^{1/m} \quad \text{where } m = m(a) . \tag{2.1.1}$$

(Remember that this definition is independent of the choice of m in $\mathbb{Z}^+$ such that a^m is in A - see remarks following 0.2 Theorem 4.)

If A is given an order with respect to which ϕ_g is an o-automorphism (an order induced by an order of G, for example), then we have the following:-

LEMMA 1. *(i)* $\phi_g^{\#}$ *is the unique extension of* $\phi_g^{\#}$ *to an* o*-automorphism of* $A^{\#}$.

(ii) For all g *and* h *in* G , $\phi_{gh}^{\#} = \phi_g^{\#}\phi_h^{\#}$.

(iii) For all a *in* A , $\phi_a^{\#} = 1$.

Proof. *(i)* See 0.2 Theorem 4 and Conrad [3], Lemma, p. 518.

(ii) Conrad proves this in his proof of Theorem 3.1 ([3], p. 519, lines 11-12).

(iii) For all a in A , $\phi_a = 1$; so $b\phi_a^{\#} = \left(b^m\phi_a\right)^{1/m} = b$ for all b in $A^{\#}$ (where $m = m(b)$). //

Now we are ready to complete A . Let $G^{\#}$ be the (set theoretic) cartesian product, $G \times A^{\#}$, modulo the equivalence

$$(g, a) = (h, b) \text{ iff } h = gc \text{ and } b = c^{-1}a \text{ for some } c \text{ in } A . \tag{2.1.2}$$

It is easy to show that (2.1.2) *does* define an equivalence relation on $G \times A^{\#}$.

Define multiplication in $G^{\#}$ by

$$(g, a)(h, b) = \left(gh, \left(a\phi_h^{\#}\right)b\right) . \tag{2.1.3}$$

To show that this definition is independent of the choice of g and h

in G and a and b in $A^{\#}$, take any c and d in A. Then

$$\begin{aligned}(gc, c^{-1}a)(hd, d^{-1}b) &= \left(gchd, (c^{-1}a)\phi^{\#}_{hd}d^{-1}b\right) \\ &= \left(gh(h^{-1}ch)d, \left(c^{-1}\phi_h\right)\left(a\phi^{\#}_h\right)d^{-1}b\right) \quad \text{(by Lemma 1)} \\ &= \left(ghd(c\phi_h), (c\phi_h)^{-1}d^{-1}\left(a\phi^{\#}_h\right)b\right) \\ &\qquad (A, A^{\#} \text{ are abelian and } A \text{ is normal in } G) \\ &= \left(gh, \left(a\phi^{\#}_h\right)d\right) \quad \text{(by (2.1.2))} \\ &= (g, a)(h, d) \quad \text{(by (2.1.3))}.\end{aligned}$$

So the definition of multiplication is satisfactory.

Associativity can be verified directly, $(1, 1)$ is an identity for $G^{\#}$ and an inverse of (g, a) is $\left(g^{-1}, a^{-1}\phi^{\#}_{g^{-1}}\right)$. So $G^{\#}$ is a group. The map $g \mapsto (g, 1)$ embeds G in $G^{\#}$ and, since $(a, 1) = (1, a)$ for all a in A and since, for b in $A^{\#}$, the map $b \mapsto (1, b)$ embeds $A^{\#}$ in $G^{\#}$, we have the following:-

THEOREM 1. *The embedding of G into $G^{\#}$ given above completes the normal, abelian, isolated subgroup, A, of G. Furthermore, $G^{\#}/A^{\#}$ is isomorphic to G/A.*

Proof. It remains to prove the latter statement. $A^{\#}$ is normal in $G^{\#}$ because, for all (g, a) and $(1, b)$ in $G^{\#}$,

$$\begin{aligned}(g, a)^{-1}(1, b)(g, a) &= \left(g^{-1}, a^{-1}\phi^{\#}_{g^{-1}}\right)\left(g, \left(b\phi^{\#}_g\right)a\right) \\ &= \left(1, a^{-1}\left(b\phi^{\#}_g\right)a\right) \\ &= \left(1, b\phi^{\#}_g\right).\end{aligned}$$

We merely observe that the obvious mapping, $(g, a)A^{\#} \mapsto gA$, is the required isomorphism of $G^{\#}/A^{\#}$ onto G/A. //

In order to discard the supposition that A is isolated, we need the following:-

LEMMA 2. *Let A be an abelian subgroup of the O-group, G. Then the isolated closure, $I(A)$, of A in G is an abelian subgroup of G. If, in addition, A is normal, then $I(A)$ is normal.*

Proof. Let $B = \{g \in G : g^m \in A \text{ for some } m \text{ in } \mathbb{Z}^+\}$. We show that B is an abelian subgroup of G and that $B = I(A)$. Take g and h in B and let m and n belong to $\mathbb{Z}^+$ such that g^m and h^n are in A. Then $[g^m, h^n] = 1$, and so $[g, h] = 1$ (see Fuchs [7], p. 38). Hence, $(gh^{-1})^{mn} = g^{mn}h^{-nm}$ which belongs to A. So gh^{-1} is in B and we have shown that B is an abelian subgroup of G.

To show that $B = I(A)$, take g in G such that g^n is in B for some n in $\mathbb{Z}^+$. Then there is m in $\mathbb{Z}^+$ such that $g^{nm} = (g^n)^m$ is in A; so g is in B. That is B is an isolated subgroup of G and, since $A \leq B$, it follows that $I(A) \leq B$. For all g in $G \backslash I(A)$, g^n is in $G \backslash I(A)$ and, hence, in $G \backslash A$ for all n in $\mathbb{Z}^+$ (because $I(A)$ is isolated and $A \leq I(A)$); so g is in $G \backslash B$ and it follows that $B \leq I(A)$.

Now suppose A is normal. We show that B is normal. Take b in B and g in G, and let b^m belong to A for m in $\mathbb{Z}^+$. Then $(g^{-1}bg)^m = g^{-1}b^mg$ is in A - so $g^{-1}bg$ is in B and, hence, B is normal. //

We are now in a position to prove:-

THEOREM 2. *If G is an O-group with normal, abelian subgroup, A, then A can be completed.*

Proof. Take an abelian d-closure, $I(A)^{\#}$, of $I(A)$ and let $G^{\#}$ be $G \times I(A)^{\#}$ modulo the appropriate equivalence (*cf.* (2.1.2)) and with the appropriate multiplication $\left(cf.\ (2.1.3)\right)$. Then by Theorem 1, the embedding G into $G^{\#}$ completes $I(A)$ and, hence, completes A. //

[Observe that if A is not isolated, then (in view of our future requirements) $I(A)$ must be used in the construction of $G^{\#}$. Otherwise, there would be g in $G\backslash A$, a in $A^{\#}\backslash A$ and m in Z^{+} such that $g^{m} = a^{m}$ is in A. That is, $(g, 1)^{m} = (1, a)^{m}$ while $(g, 1) \neq (1, a)$, an impossible situation in an O-group (see Fuchs [7], p. 37, Proposition 9). Since we want to be able to order $G^{\#}$, such obvious hindrances must be removed.]

2.2. An order for $G^{\#}$

2.2.1. Take an O-group, G, with normal, abelian subgroup, A. We suffer no loss of generality by supposing that A is isolated, so embed G in $G^{\#}$ to complete A as in Section 2.1. Now take any order, $\leq$, of G. For (g, a) in $G^{\#}$, define

$$(g, a) > 1 \text{ if, and only if, } g^{m}a^{m} > 1 \text{ in } G, \text{ where } m = m(a). \quad (2.2.1)$$

We must show that this definition is satisfactory in two senses. First, we must show that if a^{m} and a^{n} are in A, then $g^{m}a^{m} > 1$ implies $g^{n}a^{n} > 1$, and, second, that if $(g, a) > 1$, then $\left(gb, b^{-1}a\right) > 1$ for all b in A. (There is, of course, another sense in which this definition has to be satisfactory. Namely, that (2.2.1) makes $G^{\#}$ an o-group. This we show in due course.) We need the following:-

LEMMA 3. *Let* $(V, \leq)$ *be an* o*-group. Take* v, w *in* V *and* s *in* Z^{+}. *Then*

(i) $v^{s}w^{s} \leq (vw)^{s} \leq w^{s}v^{s}$ *or* $w^{s}v^{s} \leq (vw)^{s} \leq v^{s}w^{s}$, *and*

(ii) $vw > 1$ *if, and only if,* $v^{s}w^{s} > 1$.

These results are well known and we shall omit the proof. *(i)* is used in Fuchs [8], p. 84, lines 10-12, and Chehata [1], Lemma 9, says something similar. *(ii)* provides justification for trying a definition like (2.2.1).

Now take a in $A^{\#}$, g in G and m, n in Z^{+} such that a^m, a^n are in A. A standard euclidean algorithm argument shows that m and n are multiples of k in Z^{+}, where k is the smallest positive integer such that a^k is in A. Let $m = rk$ and $n = sk$ where r, s are in Z^{+}. Then

$$\begin{aligned} g^m a^m > 1 &\Rightarrow g^{rk} a^{rk} > 1 \\ &\Rightarrow g^k a^k > 1 \quad \text{(by Lemma 3 } (ii)\text{)} \\ &\Rightarrow g^{sk} a^{sk} > 1 \quad \text{(by Lemma 3 } (ii)\text{)} \\ &\Rightarrow g^n a^n > 1 . \end{aligned}$$

Now take $(g, a) > 1$ in $G^{\#}$ and any b in A. By (2.2.1), $(gb, b^{-1}a) > 1$ if, and only if, $(gb)^m (b^{-1}a)^m > 1$ in G where $m = m(b^{-1}a)$. Since b is in A, we may choose $m(b^{-1}a) = m(a)$. So, we must show that $(gb)^m b^{-m} a^m > 1$ (equivalently $b^{-m} a^m (gb)^m > 1$), knowing that $g^m a^m > 1$ (equivalently $a^m g^m > 1$), where $m = m(a) = m(b^{-1}a)$. Suppose $gb \geq bg$ in G. Then $(gb)^m \geq b^m g^m$ (Lemma 3 *(i)*). So,

$$\begin{aligned} b^{-m} a^m (gb)^m &\geq b^{-m} a^m b^m g^m \\ &= a^m g^m \quad (A \text{ abelian}) \\ &> 1 . \end{aligned}$$

Similarly, if $gb < bg$ in G, then $(gb)^m b^{-m} a^m > 1$. Hence, definition (2.2.1) makes sense. To show that (2.2.1) makes $G^{\#}$ an o-group, we need the following:-

LEMMA 4. *For* (g, a) *in* $G^{\#}$, $(g, a)^{-1} > 1$ *if, and only if,* $g^m a^m < 1$ *in* G, *where* $m = m(a)$.

Proof. Observe that $\left(a\phi_g^{\#}\right)^m = a^m\phi_g$ is in A. So, we can always choose $m\left(a\phi_g^{\#}\right) = m(a)$ (and this choice we shall make in the following argument). Since $(g, a)^{-1} = \left(g^{-1}, a^{-1}\phi_{g^{-1}}^{\#}\right)$, it follows (by (2.2.1)) that $(g, a)^{-1} > 1$ if, and only if, $g^{-m}\left(a^{-1}\phi_{g^{-1}}^{\#}\right)^m > 1$ in G. Now

$$g^{-m}\left(a^{-1}\phi_{g^{-1}}^{\#}\right)^m = g^{-m}ga^{-m}g^{-1} = g^{-(m-1)}\left(g^m a^m\right)^{-1}g^{m-1} \; ; \text{ whence,}$$

$(g, a)^{-1} > 1$ if, and only if, $\left(g^m a^m\right)^{-1} > 1$ in G and the result follows. //

Now we show that (2.2.1) satisfies *(i)*, *(ii)*, *(iii)* and *(iv)* of 0.2 Theorem 1.

(i) Take $(g, a) > 1$ with $m = m(a)$.

So $g^m a^m > 1$ in G; hence $g^m a^m \not< 1$, and so $(g, a)^{-1} \not> 1$ (by Lemma 4).

(ii) Take any $(g, a) \neq 1$ in $G^{\#}$ with $m = m(a)$.

If $g^m a^m > 1$ in G, then $(g, a) > 1$.

If $g^m a^m < 1$ in G, then $(g, a)^{-1} > 1$ (Lemma 4).

Now $g^m a^m \neq 1$ by the following argument:-

$$\begin{aligned} g^m a^m = 1 &\Rightarrow g^m = a^{-m} \\ &\Rightarrow g^m \in A \\ &\Rightarrow g \in A \quad (A \text{ isolated in } G) \\ &\Rightarrow g = a^{-1} \quad (A^{\#} \text{ an } R\text{-group}) \\ &\Rightarrow (g, a) = 1 . \end{aligned}$$

(iii) Take $(g, a) > 1$ and $(h, b) > 1$ in $G^{\#}$. Since $a^{m(a)m(b)}$ and $b^{m(b)m(a)}$ are in A, we may choose (and shall choose in the following argument) $m = m(a) = m(b)$. By definitions (2.1.3) and (2.2.1), we have

$(g, a)(h, b) > 1$ if, and only if, $(gh)^m(h^{-1}a^m h)b^m > 1$ in G.

Suppose $gh \geq hg$. Then, transforming each side by h, we have $(h^{-1}gh)h \geq h(h^{-1}gh)$. So,

$$(gh)^m = (h(h^{-1}gh))^m \geq h^m(h^{-1}gh)^m = h^m(h^{-1}g^m h) .$$

Hence

$$\begin{aligned}(gh)^m(h^{-1}a^m h)b^m &\geq h^m(h^{-1}g^m h)(h^{-1}a^m h)b^m \\ &= b^{-m}[(b^m h^m)h^{-1}(g^m a^m)h]b^m \\ &> 1 \quad (\text{because } b^m h^m > 1 \text{ and } g^m a^m > 1 \text{ in } G).\end{aligned}$$

Similarly, if $gh < hg$, we have

$$\begin{aligned}(gh)^m(h^{-1}a^m h)b^m &> (h^{-1}g^m h)h^m b^m(h^{-1}a^m h) \\ &= h^{-1}a^{-m}h[h^{-1}(a^m g^m)h(h^m b^m)]h^{-1}a^m h \\ &> 1 .\end{aligned}$$

(iv) Take $(g, a) > 1$ and (h, b) in $G^{\#}$. Since $(h, b) = (h, 1)(1, b)$ and since $(uv)^{-1}w(uv) = v^{-1}(u^{-1}wu)v$ is an identity for groups, we may show, separately, that $(h, 1)^{-1}(g, a)(h, 1) > 1$ and $(1, b)^{-1}(g, a)(1, b) > 1$. Now $(h, 1)^{-1}(g, a)(h, 1) = \left(h^{-1}gh, a\phi_h^{\#}\right)$. Setting $m = m(a) = m\left(a\phi_h^{\#}\right)$, we have $\left(h^{-1}gh, a\phi_h^{\#}\right) > 1$ if, and only if, $h^{-1}g^m h\left(a^m\phi_h\right) > 1$ in G, where $h^{-1}g^m h\left(a^m\phi_h\right) = h^{-1}(g^m a^m)h > 1$ in G. So $(h, 1)^{-1}(g, a)(h, 1) > 1$. Now $(1, b)^{-1}(g, a)(1, b) = \left(g, \left(b\phi_g^{\#}\right)^{-1}ab\right)$. Setting $m = m(a) = m(b)$, we have $\left(g, \left(b\phi_g^{\#}\right)^{-1}ab\right) > 1$ if, and only if,

$g^m(g^{-1}b^{-m}g)a^m b^m > 1$ in G, the latter being equivalent to $a^m g^m[g, b^m] > 1$ in G. If $[g, b^m] \geq 1$ in G, then $a^m g^m[g, b^m] \geq a^m g^m > 1$. Suppose $[g, b^m] < 1$ in G. Now

$$[g, b^m] < 1 \Rightarrow b^{-m}gb^m < g$$

$$\Rightarrow b^{-m}g^{m-1}b^m = (b^{-m}gb^m)^{m-1} \leq g^{m-1}$$

(only if $m = 1$ does equality hold).

So,

$$g^m a^m > 1 \Rightarrow g^m > a^{-m}$$

$$\Rightarrow g^{m-1} > a^{-m}g^{-1}$$

$$\Rightarrow b^{-m}g^{m-1}b^m > b^{-m}(a^{-m}g^{-1})b^m$$

$$\Rightarrow g^{m-1} > b^{-m}(a^{-m}g^{-1})b^m \quad (\text{since } g^{m-1} \geq b^{-m}g^{m-1}b^m)$$

$$\Rightarrow g^m > a^{-m}[b^m, g]$$

$$\Rightarrow a^m g^m[g, b^m] > 1 .$$

So, $(1, b)^{-1}(g, a)(1, b) > 1$ and, hence, $(h, b)^{-1}(g, a)(h, b) > 1$.

So, $(G^\#, \leq)$ is an o-group. Since the order of $G^\#$ extends that of G (that is, $(g, 1) > 1$ in $G^\#$ if, and only if, $g > 1$ in G), we have:-

THEOREM 3. *Let G be an O-group with normal, abelian subgroup, A. Then A can be completed by o^*-embedding G in an O-group, $G^\#$. If $A^\#$ is the d-closure of (the image under the embedding of) A, then $G^\#/A^\#$ is isomorphic to G/A.*

As a corollary, we have the sufficient condition mentioned earlier.

COROLLARY 1. *Let G be an O-group, let a be an element of G and let n be in Z^+. If $\{a\}^G$ (the normal closure of $\{a\}$ in G) is abelian, then G can be o^*-embedded in an O-group, H, in which there is a solution to the equation $x^n = a$.*

Observe that $\{a\}^G$ is abelian if, and only if, $[g, a, a] = 1$ for all g in G. So Corollary 1 can be rephrased as:-

COROLLARY 1'. *Let G, a and n be as in Corollary 1. If $[g, a, a] = 1$ for all g in G, then G can be o*-embedded in an O-group, H, in which there is a solution to the equation $x^n = a$.*

2.2.2. At this stage, we ask what happens if for *all* a in G, $[g, a, a] = 1$ for all g in G? This will occur if, for instance, G is nilpotent of class 2. In fact, $[g, a, a] = 1$ for all g, a in G if, and only if, G is a 2-engel group. (This is a definition, not a theorem.) However, G is torsion-free, and since torsion-free 2-engel groups are nilpotent of class 2 (Schenkman, [22], p. 208), we shall simply consider the case where G is nilpotent of class 2. (We mention again that it is known that (locally) nilpotent O-groups have (locally) nilpotent, orderable d-closures. Whether or not such d-closures are d^*-closures does not appear to be known.)

Writing, as usual, $\underline{\underline{N}}_2$ for the variety of groups which are nilpotent of class at most 2, we have the following:-

THEOREM 4. *Let G, A, $G^{\#}$ and $A^{\#}$ be as in Theorem 3 with the additional condition that G is in $\underline{\underline{N}}_2$. Then $G^{\#}$ is in $\underline{\underline{N}}_2$.*

Proof. Take any (g_j, a_j) in $G^{\#}$, $j = 1, 2, 3$.

$$[(g_1, a_1), (g_2, a_2)] = \left([g_1, g_2], \left(a_1^{-1}\phi^{\#}_{[g_1, g_2]}\right)\left(a_2^{-1}\phi^{\#}_{g_2^{-1}g_1g_2}\right)\left(a_1\phi^{\#}_{g_2}\right)a_2\right)$$

$$= \left([g_1, g_2], a_1^{-1}\left(a_2^{-1}\phi^{\#}_{g_1}\right)\left(a_1\phi^{\#}_{g_2}\right)a_2\right)$$

since, for all a in A and g, h in G,

$$a\phi_{[g,h]} = a \text{ and } a\phi_{h^{-1}gh} = a\phi_{[h,g^{-1}]g} = a\phi_g \quad (G \text{ is in } \underline{\underline{N}}_2).$$

So,

$$[(g_1, a_1), (g_2, a_2), (g_3, a_3)] = ([g_1, g_2, g_3], a')$$

$$= (1, a') \quad (G \text{ is in } \underline{\underline{N}}_2)$$

where

$$a' = a_1\left(a_1^{-1}\phi^{\#}_{g_2}\right)\left(a_1^{-1}\phi^{\#}_{g_3}\right)\left(a_1\phi^{\#}_{g_2g_3}\right)a_2^{-1}\left(a_2\phi^{\#}_{g_1}\right)\left(a_2\phi^{\#}_{g_3}\right)\left(a_2^{-1}\phi^{\#}_{g_1g_3}\right) .$$

Take any a in A and g, h in G, and, remembering that G is in $\underline{\underline{N}}_2$, we have

$$\begin{aligned} a\left(a^{-1}\phi_g\right)\left(a^{-1}\phi_h\right)\left(a\phi_{gh}\right) &= a\left(a^{-1}[a^{-1}, g]\right)\left(a^{-1}[a^{-1}, h]\right)(a[a, gh]) \\ &= [a^{-1}, gh][a, gh] \\ &= 1 . \end{aligned}$$

Setting $m = m(a_1) = m(a_2)$, it follows that $(a')^m = 1$; whence $a' = 1$ ($A^{\#}$ is torsion-free). So, $G^{\#}$ is in $\underline{\underline{N}}_2$. //

Since G in $\underline{\underline{N}}_2$ implies that $\{a\}^G$ is abelian for all a in G, we have, by the usual transfinite arguments, the following theorem.

THEOREM 5. *If G is in $\underline{\underline{N}}_2$ and is orderable, then G has a d^*-extension (and, hence, a d-closure, H) which is in $\underline{\underline{N}}_2$.*

We have the following analogue to 2.1 Lemma 1 *(i)*.

LEMMA 5. *Let G and H be as in Theorem 5 above. Then any O-automorphism of G extends uniquely to an O-automorphism of H.*

Proof. Observing that for all h in H, there is m in Z^+ such that h^m is in G (see Kurosh [17], p. 256, paragraph 3), we define $\phi' : H \to H$ by $h\phi' = \left(h^m\phi\right)^{1/m}$, where ϕ is an O-automorphism of G. We shall omit proofs. //

This lemma suggests that, in the construction of $G^{\#}$, we ought to be able to make A nilpotent of class 2 rather than abelian. My efforts in this direction have, to date, been unsuccessful.

2.2.3. We conclude this section with some comments on the theorem of Kopytov [16] (see also Fuchs [8], p. 83, Satz 15) and the theorem of Conrad

[3] (p. 518, Theorem 3.1).

The construction of $G^{\#}$ and the definition (2.2.1) of order are, essentially, due to Kopytov. His theorem is our Theorem 3 with "normal, abelian subgroup" replaced by "centre" (and with the addition that if G is an O^*-group, then $G^{\#}$ is an O^*-group. We shall make the corresponding addition to Theorem 3 in Section 2.3).

Conrad's theorem is less general in that the normal, abelian subgroup, A, is required to be contained in a normal, abelian subgroup of G which is relatively convex in G (that is, convex in at least one order of G). The existence of O-groups with maximal, normal, abelian subgroups which are not relatively convex shows that the theorem of Conrad is not the most general. (In [14], question 1.44, Kokorin asks whether a maximal, normal, abelian subgroup of an O-group is relatively convex. The answer given is "not always" and the reference is to Smirnov [23].)

2.3 Further properties of the embedding $G \to G^{\#}$

Throughout this section, G, A, $G^{\#}$ and $A^{\#}$ are as in section 2.2. (Remember that we are supposing A to be isolated in G.) The results of this section (a series of diverse theorems) are presented with the thought that, while most are somewhat incidental to the main theme of this thesis, they are of some interest in their own right.

2.3.1. Denote by $\Omega(G)$ the set of all full orders of G. Identifying an order, $\leq$, of G by its positive cone, P, we write $P^{\#}$ for the order of $G^{\#}$ given by (2.2.1). That is, $(g, a) \in P^{\#}$ if, and only if, $g^m a^m \in P$, where $m = m(a)$.

THEOREM 6. *The mapping $P \mapsto P^{\#}$ is a one-to-one mapping from $\Omega(G)$ onto $\Omega(G^{\#})$.*

Proof. Since $P \subseteq P^{\#}$ $\left(\text{that is, } P^{\#} \cap G = P\right)$ for all P in $\Omega(G)$ (see remark preceding 2.2 Theorem 3), it follows that $P_1 \neq P_2$ in $\Omega(G)$ implies $P_1^{\#} \neq P_2^{\#}$ in $\Omega\left(G^{\#}\right)$; so the mapping is one-to-one. Now take any P' in $\Omega\left(G^{\#}\right)$ and let $P = P' \cap G$. We show that $P^{\#} = P'$, whence the mapping is onto. If (g, a) is in $P^{\#}$, then $g^m a^m$ is in P $\left(m = m(a)\right)$ by (2.2.1). Since $\left(g^m a^m, 1\right) = \left(g^m, a^m\right)$ in $G^{\#}$, and since $P = P' \cap G$, it follows that $\left(g^m, a^m\right)$ is in P' . Since

$$\left(g^m, a^m\right) = \left(g^m, 1\right)\left(1, a^m\right) = (g, 1)^m (1, a)^m$$

in $G^{\#}$, 2.2 Lemma 3 *(ii)* implies that $(g, a) = (g, 1)(1, a)$ is in P' . So, $P^{\#} \subseteq P'$ and, since the orders are linear, it follows that $P^{\#} = P'$. //

2.3.2. Here we present some properties of A preserved under the mapping.

A subgroup of an O-group is said to be *relatively* (respectively *absolutely*) *convex* if the subgroup is convex in at least one order (respectively all orders) of the group.

THEOREM 7. *A is relatively (respectively absolutely) convex in G if, and only if, $A^{\#}$ is relatively (respectively absolutely) convex in $G^{\#}$.*

Proof. A is relatively convex in G if, and only if, G/A is an O-group (*cf.* Fuchs [7]), p. 21 Theorem 7. The corresponding result for $A^{\#}$ and $G^{\#}$, plus the fact that $G^{\#}/A^{\#}$ is isomorphic to G/A (2.2 Theorem 3), gives the relatively convex part of the theorem.

Now suppose A is absolutely convex in G and let $\leq^{\#}$ be any order of $G^{\#}$ $\left(\text{where } \leq^{\#}|_G \text{ is } \leq\right)$. Take $1 <^{\#} (g, a) <^{\#} (1, b)$ where (g, a) is in $G^{\#}$ and $(1, b)$ is in $A^{\#}$. Setting $m = m(a) = m(b)$, we have $1 <^{\#} (g, a)^m <^{\#} (1, b)^m$, and since $1 <^{\#} g^m a^m <^{\#} (g, a)^m$ or

$1 <^{\#} a^m g^m <^{\#} (g, a)^m$ (by Lemma 3), we have $1 < g^m a^m < b^m$ or $1 < a^m g^m < b^m$. Since a^m and b^m are in A and since A is convex, $g^m a^m$ (or $a^m g^m$) and, hence, g^m is in A. So, g is in A (since A is isolated), whence (g, a) is in $A^{\#}$. So $A^{\#}$ is convex, and since $\leq^{\#}$ was chosen arbitrarily, $A^{\#}$ is absolutely convex in $G^{\#}$.

Conversely, suppose $A^{\#}$ is absolutely convex in $G^{\#}$ and let $\leq$ be any order of G (with $\leq^{\#}$ the corresponding order of $G^{\#}$). Take $1 < g < a$ with g in G and a in A. Then
$1 <^{\#} (g, 1) <^{\#} (a, 1) = (1, a)$; so $(g, 1)$ is in $A^{\#}$ (since $A^{\#}$ is convex in $G^{\#}$) and, hence, g is in A. As before, A is absolutely convex in G. //

A normal subgroup, V, of a group, W, is said to be *strongly isolated in* W if for $w, w_1, w_2, \ldots, w_k$ in W,

$w_1^{-1}ww_1w_2^{-1}ww_2 \ldots w_k^{-1}ww_k \in V$ implies $w \in V$.

THEOREM 8. *A is strongly isolated in G if, and only if, $A^{\#}$ is strongly isolated in $G^{\#}$.*

Proof. The above definition says, in effect, that A is strongly isolated in G if, and only if, G/A is an *NGP*-group. $G^{\#}/A^{\#}$ is isomorphic to G/A, so the result follows. //

THEOREM 9. *A is central in G if, and only if, $A^{\#}$ is central in $G^{\#}$.*

Proof. (This is essentially Kopytov's Theorem.) Suppose A is central in G. Then $\phi_g^{\#} = 1$ for all g in G. So, for all (g, a) in $G^{\#}$ and $(1, b)$ in $A^{\#}$, we have

$$(g, a)(1, b) = (g, ab) = (g, ba) = \left(g, \left(b\phi_g^{\#}\right)a\right) = (1, b)(g, a) .$$

Conversely, suppose $A^\#$ is central in $G^\#$. Then for all g in G and a in A, we have $(ga, 1) = (g, 1)(a, 1) = (a, 1)(g, 1) = (ag, 1)$; whence $ga = ag$. //

2.3.3. Now for some properties of G which $G^\#$ inherits. The first is the promised addition to 2.2 Theorem 3.

We need the following (probably well-known) lemma.

LEMMA 6. *Let* $(V, \leq)$ *be a partially ordered group. Then for* v, w *in* V *and* s *in* $\mathbb{Z}^+$, $vw > 1$ *implies* $v^s w^s > 1$.

Proof. Clearly, the lemma is true for $s = 1$. Suppose it is true for all $s < t$ and take $vw > 1$. Then

$$v^t w^t = v\left(v^{t-1}w^{t-1}\right)w > v1w = vw > 1 . \quad //$$

THEOREM 10. *If* G *is an* O^**-group, then* $G^\#$ *is an* O^**-group.*

Proof. Let $P^\#$ be (the positive cone of) a partial order of $G^\#$ and let $P = P^\# \cap G$. Then P is a partial order of G and, since G is an O^*-group, there is a full order, Q, of G such that $P \subseteq Q$. Let $Q^\#$ be the full order of $G^\#$ induced by Q (that is, $Q^\#$ is obtained from Q by (2.2.1)). We show that $P^\# \subseteq Q^\#$.

$$\begin{aligned}
(g, a) \in P^\# &\Rightarrow \left(g^m, a^m\right) \in P^\# \text{ where } m = m(a) \text{ (by Lemma 6)}\\
&\Rightarrow g^m a^m \in P \quad \left(\text{since } P = P^\# \cap G\right)\\
&\Rightarrow g^m a^m \in Q\\
&\Rightarrow (g, a) \in Q^\# \quad \left(\text{by (2.2.1)}\right). \quad //
\end{aligned}$$

The next two theorems are required for Chapter 3.

THEOREM 11. *If* G *is metabelian and* $A = I(G')$, *then* $G^\#$ *is metabelian and* $A^\# = I\left(\left(G^\#\right)'\right)$.

Proof. Since $G' \leq A$ we have G/A abelian. So $G^\#/A^\#$ is abelian, whence $G^\#$ is metabelian. So, $\left(G^\#\right)' \leq A^\#$ and, since $A^\#$ is divisible (hence isolated in $G^{\#}$), $I((G^{\#})') \le A^{\#}$. Take any $(1, a)$ in $A^{\#}$. Setting $m = m(a)$, we have a^m in $A = I(G')$. So, a^{mn} is in G' for some n in Z^+ (because G' is abelian - see 2.1 Lemma 2). Since $G \le G^{\#}$, it follows that $G' \le (G^{\#})'$ and so $(1, a^{mn})$ is in $(G^{\#})'$. That is, $(1, a)^{mn}$ is in $(G^{\#})'$; so $(1, a)$ is in $I((G^{\#})')$. Hence $A^{\#} = I((G^{\#})')$. //

THEOREM 12. *If* G *splits over* A, *then* $G^{\#}$ *splits over* $A^{\#}$.

Proof. Suppose G splits over A. Then there is a subgroup, B, of G with B isomorphic to G/A, $A \cap B = \{1\}$ and $G = \mathrm{gp}(A, B)$ (see, for example, Kurosch [17], p. 149). Let $B_{\#} = \{(b, 1) \in G^{\#} : b \in B\}$. Then $B_{\#}$ is (naturally) isomorphic to B and, hence, to $G^{\#}/A^{\#}$. To complete the proof, we show that $A^{\#} \cap B_{\#} = \{1\}$ and that $G^{\#} = \mathrm{gp}\left(A^{\#}, B_{\#}\right)$.

If (g, a) is in $A^{\#} \cap B_{\#}$, then $(g, a) = (1, a')$ and $(g, a) = (b, 1)$ for some a' in $A^{\#}$ and b in B. Now $(b, 1) = (1, a')$ is only possible if b and a' are in A see (2.1.2). Since b is in A only if $b = 1$, it follows that $A^{\#} \cap B_{\#} = \{1\}$. Now take any (g, a) in $G^{\#}$. We have $g = ba'$ for some b in B and a' in A, whence $(g, a) = (ba', a) = (b, a'a) = (b, 1)(1, a'a)$. So, $G^{\#} = \mathrm{gp}\left(A^{\#}, B_{\#}\right)$. //

The final result of this section is the theorem of Terehov [24] referred to earlier.

A group, V, is an S-ext-*group* if every full order of every subgroup of V can be extended to a full order for V. The following characterization of S-ext-groups is due to Kargapalov [13]. (The theorem appears also in Minassian [20].)

THEOREM 13. *W is an S-ext-group if, and only if, W has a normal, abelian subgroup, V, such that W/V is abelian and for arbitrary elements v in V and w in $W\backslash V$, there are positive integers, $s \neq t$, such that $w^{-1}v^{s}w = v^{t}$.*

We shall call V the *characterizing subgroup* of the S-ext-group, W.

THEOREM 14. (Terehov [24]). *If G is an S-ext-group with characterizing subgroup, A, then $G^{\#}$ is an S-ext-group with characterizing subgroup, $A^{\#}$.*

Proof. $A^{\#}$ is normal and abelian and $G^{\#}/A^{\#}$, being isomorphic to G/A, is abelian. Take any $(1, b)$ in $A^{\#}$ and (g, a) in $G^{\#}\backslash A^{\#}$ (so, g is in $G\backslash A$). Take $m = m(b)$ and $r \neq s$ in $\mathbf{Z}^{+}$ such that $g^{-1}b^{mr}g = b^{ms}$. Then

$$\begin{aligned}(g, a)^{-1}(1, b)^{r}(g, a) &= \left(1, b^{r}\phi_{g}^{\#}\right)\\ &= \left(1, \left(g^{-1}b^{rm}g\right)^{1/m}\right)\\ &= \left(1, \left(b^{sm}\right)^{1/m}\right)\\ &= (1, b)^{s} . \qquad //\end{aligned}$$

CHAPTER 3

THE METABELIAN CASE

3.0 Introduction

Kokorin asks ([14], question 1.60), "Can a metabelian, orderable group be embedded in a divisible (metabelian) orderable group?" The next two chapters report progress made on the problem.

Given a metabelian, orderable group, G, (denoted by *MO-group* in the sequel) with $A = I(G')$ and $\Phi = G/A$, a possible approach to the problem is as follows:-

(1) embed G in a split extension, G_1, of A_1 by Φ where G_1 is an *MO*-group and A_1 is abelian,

(2) embed G_1 in an *MO*-group, G_2, which is a split extension of a divisible, abelian group, A_2, by Φ,

(3) embed G_2 in an *MO*-group, G_3, which is a split extension of a divisible, abelian group, A_3, by $\Phi^{\#}$ (the abelian d-closure of Φ), and

(4) embed G_3 in an *MO*-group, G_4, which is a split extension of a divisible, abelian group, A_4, by $\Phi^{\#}$ such that the endomorphism, $1 + \phi + \ldots + \phi^{n-1}$, of A_4 is onto for all ϕ in $\Phi^{\#}$ and n in Z^+. (Here we have identified ϕ in $\Phi^{\#}$ with the automorphism it induces naturally on A_4 and have used the natural ring structure of the set of endomorphisms of an abelian group (see, for example, Fuchs [6], p. 210 ff).)

If it is possible to accomplish all of steps (1), (2), (3) and (4), then the resulting *MO*-group, G_4, is divisible (3.3 Theorem 4). At present, I have solved (1), (2) and (4) (Sections 3.1, 3.2 and 3.3 respectively) while (3) remains open (Chapter 4), although I expect a solution to (3) is not far off.

It would be nice, of course, if the embeddings all proved to be o^*-embeddings. The embedding used in (2), being that of Chapter 2, is an o^*-embedding, while the embeddings of (1) and (4) have defied my efforts to prove (or disprove) them to be o^*-embeddings.

3.1 Embedding in a split extension

3.1.1. Let G be an *MO*-group with $A = I(G')$ and $\Phi = G/A$. We shall suppose that G is non-abelian; so $\{1\} \neq A$ and (by 2.1 Lemma 2) $A \neq G$. Furthermore, Φ is torsion-free (since A is isolated) and abelian, and hence orderable (see Fuchs [7], p. 36). To embed G in a split extension of A_1 by Φ (with A_1 abelian) we use the construction described in Zassenhaus ([25], pp. 133-134). This construction is presented here with proofs omitted.

Let $\tau : \Phi \to G$ be a transversal function. That is, $\phi(\tau\varepsilon) = \phi$ for all ϕ in Φ, where $\varepsilon : G \to \Phi$ is the natural epimorphism. To avoid confusion later, we write τ on the left (giving a rather messy $(\tau(\phi))\varepsilon = \phi$ which, happily, we do not encounter again). In particular, we choose $\tau(1) = 1$.

Define $f : \Phi \times \Phi \to A$ by

$$f(\phi, \chi) = \tau(\phi\chi)^{-1}\tau(\phi)\tau(\chi) \quad \text{for all } \phi, \chi \text{ in } \Phi . \tag{3.1.1}$$

Then G is isomorphic to the group which is the set, $\Phi \times A$, with multiplication given by

$$(\phi, a)(\chi, b) = \left(\phi\chi, f(\phi, \chi)a^{\tau(\chi)}b\right)$$

for all ϕ, χ in Φ and a, b in A (where, as usual, for elements u and v of a group, we write $u^v = v^{-1}uv$).

For all $\phi \neq 1$ in Φ, let A_ϕ be the infinite cyclic group with generator a_ϕ (a new symbol) and let $\bar{A}$ be the restricted direct product $A \times \left(\prod_{\phi \neq 1} A_\phi\right)$. For all $\bar{a} = a\left(\prod_{\phi \neq 1} a_\phi^{m_\phi}\right)$ in $\bar{A}$ (where m_ϕ is in $\mathbb{Z}$ and the product is finite) and χ in Φ, define

$$\bar{a}^{\tau(\chi)} = a^{\tau(\chi)}\left(\prod_{\phi \neq 1} \left(a_\phi^{\tau(\chi)}\right)^{m_\phi}\right) \tag{3.1.2}$$

where (setting $a_1 = 1$) we define

$$a_\phi^{\tau(\chi)} = a_{\phi\chi} a_\chi^{-1} f(\phi, \chi)^{-1} . \tag{3.1.3}$$

Then the mapping $\bar{a} \mapsto \bar{a}^{\tau(\chi)}$ of $\bar{A}$ into itself is an automorphism of $\bar{A}$ for all χ in Φ. (In the sequel, we shall refer to this automorphism as "*the automorphism* $\tau(\chi)$".) Furthermore,

$$\left(\bar{a}^{\tau(\chi)}\right)^{\tau(\psi)} = \bar{a}^{\tau(\chi\psi)} \quad \text{for all } \chi, \psi \text{ in } \Phi . \tag{3.1.4}$$

Let $\bar{G} = \Phi \times \bar{A}$ with multiplication given by

$$(\phi, \bar{a})(\chi, \bar{b}) = \left(\phi\chi, f(\phi, \chi)\bar{a}^{\tau(\chi)}\bar{b}\right) . \tag{3.1.5}$$

Then $\bar{G}$ is a split extension of $\bar{A}$ by Φ and G can be embedded in $\bar{G}$.

3.1.2. To show that $\bar{G}$ is an O-group, we order $\bar{A}$ in such a way that, for all ϕ in Φ, the automorphism $\tau(\phi)$ is an o-automorphism. It follows that the order of $\bar{G}$ given by $(\phi, \bar{a}) > 1$ if, and only if, $\phi > 1$, *or* $\phi = 1$ and $\bar{a} > 1$, makes $\bar{G}$ an o-group. (Here, the order of Φ is arbitrary but fixed.)

At this stage, we point out that any order of G in which A is convex can be extended to an order of $\bar{G}$. The fate of other orders of G remains undecided at this stage.

Throughout this section, we shall use the following standard form for elements of $\bar{A}$.

Choosing any (full) order, $\leq$, of Φ, we write

$$\bar{a} = a\left(\prod_{j=1}^{r} a_j^{m_j}\right)$$

where

either $\prod_{j=1}^{r} a_j^{m_j} \neq 1$ (so $r \geq 1$) with m_j in $\mathbb{Z}\backslash\{0\}$ and

$a_j = a_{\phi_j}$ for $j = 1, 2, \ldots, r$, and $\phi_1 < \phi_2 < \ldots < \phi_r$ in $\Phi\backslash\{1\}$,

or $\prod_{j=1}^{r} a_j^{m_j} = 1$.

We distinguish these cases by saying, for $\bar{a}$ in $\bar{A}$, that *either* $r \geq 1$ (in which case $\bar{a}$ is not in A), *or* $r = 0$.

We order $\bar{A}$ as follows:-

Take any order, $\leq$, of A and for $\bar{a} = a\left(\prod_{j=1}^{r} a_j^{m_j}\right)$ in $\bar{A}$, define $\bar{a} > 1$ if, and only if,

$$\left.\begin{aligned}
&(1)\quad r \geq 1\,,\ \phi_r > 1 \textit{ and } m_r > 0\,, \textit{ or}\\
&(2)\quad r \geq 1\,,\ \phi_r < 1 \textit{ and } \sum_{j=1}^{r} m_j < 0\,, \textit{ or}\\
&(3)\quad r \geq 1\,,\ \phi_r < 1\,,\ \sum_{j=1}^{r} m_j = 0 \textit{ and } m_r > 0\,, \textit{ or}\\
&(4)\quad r = 0 \textit{ and } a > 1 \textit{ in } A\,.
\end{aligned}\right\} \qquad (3.1.6)$$

LEMMA 1. $(\bar{A}, \leq)$ *is an o-group.*

Proof. We show that *(i)*, *(ii)* and *(iii)* of 0.2 Theorem 1 hold. (Since $\bar{A}$ is abelian, *(iv)* is obvious.)

Since $\bar{a}^{-1} = \bar{a}^{-1}\left(\prod_{j=1}^{r} a_j^{-m_j}\right)$, we have by (3.1.6) $a^{-1} > 1$ if, and only if,

$r \geq 1$, $\phi_r > 1$ and $-m_r > 0$, *or*

$r \geq 1$, $\phi_r < 1$ and $\sum_{j=1}^{r} \left(-m_j\right) < 0$, *or*

$r \geq 1$, $\phi_r < 1$, $\sum_{j=1}^{r} (-m_j) = 0$ and $-m_r > 0$, *or*

$r = 0$ and $a^{-1} > 1$ in A .

Inspection reveals that if $\bar{a} > 1$, then $\bar{a}^{-1} \ngtr 1$; and that if $\bar{a} \neq 1$, then $\bar{a} > 1$ or $\bar{a}^{-1} > 1$.

Now take $\bar{a} = a\left(\prod_{j=1}^{r} a_j^{m_j}\right) > 1$ and $\bar{b} = b\left(\prod_{k=1}^{s} b_k^{n_k}\right) > 1$ in $\bar{A}$, where $b_k = a_{\chi_k}$ for $k = 1, 2, \ldots, s$. Write $\bar{a}\bar{b} = c\left(\prod_{l=1}^{t} c_l^{p_l}\right)$, where $c_l = a_{\psi_l}$ for $l = 1, 2, \ldots, t$. Then

$$c = ab \text{ in } A \text{ and } \sum_{l=1}^{t} p_l = \sum_{j=1}^{r} m_j + \sum_{k=1}^{s} n_k . \tag{3.1.7}$$

To show that $\bar{a}\bar{b} > 1$, we consider the various possible cases for $\bar{a}$ and $\bar{b}$ (remembering that $\bar{a}\bar{b} = \bar{b}\bar{a}$, so we need not consider all cases). For convenience, we set $m = \sum_{j=1}^{r} m_j$, $n = \sum_{k=1}^{s} n_k$ and $p = \sum_{l=1}^{t} p_l$.

(1) $r \geq 1$, $\phi_r > 1$ *and* $m_r > 0$.

(1.1) $s \geq 1$, $\chi_s > 1$ *and* $n_s > 0$. Since $m_r > 0$ and $n_s > 0$, it follows that $a_r^{m_r} a_s^{n_s} \neq 1$; whence $t \geq 1$, $\psi_t = \max\{\phi_r, \chi_s\} > 1$ and $0 < p_t = m_r, m_r+n_s$ or n_s according as $\phi_r > \chi_s$, $\phi_r = \chi_s$ or $\phi_r < \chi_s$.

(1.2) $s \geq 1$, $\chi_s < 1$ *and* $n < 0$. In this case, $t \geq 1$, $\psi_t = \phi_r > 1$ and $p_t = m_r > 0$.

Cases (1.3) ($s \geq 1$, $\chi_s < 1$, $n = 0$ *and* $n_s > 0$) and (1.4) ($s = 0$ *and* $b > 1$ *in* A) yield the same conclusion as (1.2).

(2) $r \geq 1$, $\phi_r < 1$ *and* $m < 0$.

(2.1) $s \geq 1$, $\chi_s > 1$ *and* $n_s > 0$. Since $\bar{a}\bar{b} = \bar{b}\bar{a}$, this case is virtually identical to (1.2).

(2.2) $s \geq 1$, $\chi_s < 1$ *and* $n < 0$. By (3.1.7), $p < 0$, whence, in addition, $t \geq 1$ and $\psi_t \leq \max\{\phi_r, \chi_s\} < 1$. (We have $\leq$, not $=$, here because $\phi_r = \chi_s$ and $a_r^{m_r} a_s^{n_s} = 1$ is possible.)

(2.3) $s \geq 1$, $\chi_s < 1$, $n = 0$ *and* $n_s > 0$. This case is similar to (2.2).

(2.4) $s = 0$ *and* $b > 0$ *in* A . Here $t \geq 1$, $\psi_t = \phi_r > 1$ and $p = m < 0$.

(3) $r \geq 1$, $\phi_r < 1$, $m = 0$ *and* $m_r > 0$.

Cases (3.1) and (3.2) are identical to cases (1.3) and (2.3) respectively.

(3.3) $s \geq 1$, $\chi_s < 1$, $n = 0$ *and* $n_s > 0$. Since $m_r > 0$ and $n_s > 0$, it follows that $a_r^{m_r} a_s^{n_s} \neq 1$; whence $t \geq 1$, $\psi_t = \max\{\phi_r, \chi_s\} < 1$, $p = 0$ (by (3.1.7)) and $0 < p_t = m_r, m_r + n_s$ or n_s according as $\phi_r > \chi_s$, $\phi_r = \chi_s$ or $\phi_r < \chi_s$.

(3.4) $s = 0$ *and* $b > 1$ *in* A . Here $t \geq 1$, $\psi_t = \phi_r < 1$, $p = 0$ and $p_t = m_r > 0$.

(4) $r = 0$ *and* $a > 1$ *in* A .

(4.1), (4.2) and (4.3) are identical to (1.4), (2.4) and (3.4) respectively.

(4.4) $s = 0$ *and* $b > 1$ *in* A . Here $t = 0$ and $c = ab > 1$ in A .

So $\bar{a}\bar{b} > 1$ (since c, t, ψ_t and $p_1, p_2, \ldots, p_t$ always satisfy one of the conditions of (3.1.6)) and we have shown that $(\bar{A}, \leq)$ is an *o*-group. //

LEMMA 2. *$\bar{A}$ can be ordered in such a way that every automorphism,*

$\tau(\chi)$, *(as* χ *ranges over* Φ *) is an o-automorphism*

Proof. Let $\leq$ be any order of A which is induced by an order of G . Then every inner automorphism of G induces an o-automorphism of A , whence $a \mapsto a^{\tau(\chi)}$ is an o-automorphism of A for all χ in Φ . Ordering $\bar{A}$ by (3.1.6), we shall show that $\tau(\chi)$ is an o-automorphism of $\bar{A}$ for all $\chi > 1$ in Φ , whence $\tau(\chi)$ is an o-automorphism of $\bar{A}$ for *all* χ in Φ as follows:- for all $\phi \neq 1$ in Φ ,

$$\begin{aligned} a_\phi^{\tau(1)} &= a_\phi a_1^{-1} f(\phi, 1)^{-1} \quad \text{(by (3.1.3))} \\ &= a_\phi \quad (\tau(1) = 1 \text{ implies } f(\phi, 1) = 1) . \end{aligned}$$

So,

$$\begin{aligned} \left(\bar{a}^{\tau(\chi)}\right)^{\tau\left(\chi^{-1}\right)} &= \bar{a}^{\tau(1)} \quad \text{(by (3.1.4))} \\ &= \bar{a} \quad \text{for all } \bar{a} \text{ in } \bar{A} \text{ and } \chi \text{ in } \Phi . \end{aligned}$$

That is, the automorphism $\tau\left(\chi^{-1}\right)$ is the inverse of the automorphism $\tau(\chi)$ for all χ in Φ . Since the set of all o-automorphisms of an o-group forms a group under composition, it follows that $\tau\left(\chi^{-1}\right)$ is an o-automorphism whenever $\tau(\chi)$ is an o-automorphism.

Now take any $\chi > 1$ in Φ and any $\bar{a} = a\left(\prod_{j=1}^{r} a_j^{m_j}\right) > 1$ in $\bar{A}$. By (3.1.2) and (3.1.3), we have

$$\bar{a}^{\tau(\chi)} = \left(a^{\tau(\chi)} \prod_{j=1}^{r} f\left(\phi_j, \chi\right)^{-m_j}\right)\left(\left(\prod_{j=1}^{r} d_j^{m_j}\right)\left(\prod_{j=1}^{r} a_\chi^{-m_j}\right)\right)$$

where $d_j = a_{\phi_j \chi}$ for all $j = 1, 2, \ldots, r$.

Setting $\bar{a}^{\tau(\chi)} = b\left(\prod_{k=1}^{s} b_k^{n_k}\right)$ (standard form) where $b = a^{\tau(\chi)} \prod_{j=1}^{r} f\left(\phi_j, \chi\right)^{-m_j}$ and $b_k = a_{\chi_k}$ for all $k = 1, 2, \ldots, s$, we show that $\bar{a}^{\tau(\chi)} > 1$ by considering the possible cases for $\bar{a}$. (Again, we set $m = \sum_{j=1}^{r} m_j$ and $n = \sum_{k=1}^{s} n_k$.)

(1) $r \geq 1$, $\phi_r > 1$ *and* $m_r > 0$.

$\phi_1 < \phi_2 < \ldots < \phi_r$ implies $\phi_1\chi < \phi_2\chi < \ldots < \phi_r\chi$, and $1 < \phi_r$ and $1 < \chi$ imply $1 < \chi < \phi_r\chi$. So, $s \geq 1$, $\chi_s = \phi_r\chi > 1$ and $n_s = m_r > 0$.

(2) $r \geq 1$, $\phi_r < 1$ *and* $m < 0$.

In this case, $\phi_1\chi < \phi_2\chi < \ldots < \phi_r\chi < \chi$. Since $1 < \chi$, and since $\prod_{j=1}^{r} a_\chi^{-m_j} = a_\chi^{-m}$, we have $s \geq 1$, $\chi_s = \chi > 1$ and $n_k = -m > 0$.

(3) $r \geq 1$, $\phi_r < 1$, $m = 0$ *and* $m_r > 0$.

This case is less straightforward. Note that $r \geq 1$ and $\sum_{j=1}^{r} m_j = 0$ (with all $m_j \neq 0$) imply $r \geq 2$. Now

$$\bar{a}^{\tau(\chi)} = b\left(\left(\prod_{j=1}^{r} d_j^{m_j}\right)\left(\prod_{j=1}^{r} a_\chi^{-m_j}\right)\right)$$

$$= b\left(\prod_{j=1}^{r} d_j^{m_j}\right) \quad (\text{since } m = 0 \text{ }).$$

If $\phi_r\chi > 1$, then $s = r \geq 1$, $\chi_s = \phi_r\chi > 1$ and $n_s = m_r > 0$. If $\phi_r\chi < 1$, then $s = r \geq 1$, $\chi_s = \phi_r\chi < 1$, $n = m = 0$ and $n_s = m_r > 0$.

Suppose $\phi_r\chi = 1$. Then $\bar{a}^{\tau(\chi)} = b\left(\prod_{j=1}^{r-1} d_j^{m_j}\right)$. So, $s = r-1 \geq 1$ (since $r \geq 2$), $\chi_s = \phi_{r-1}\chi < \phi_r\chi = 1$ and $n = \sum_{j=1}^{r-1} m_j = m - m_r = 0 - m_r < 0$.

(4) $r = 0$ *and* $a > 1$ *in* A .

Here, $s = 0$ and $\bar{a}^{\tau(\chi)} = a^{\tau(\chi)} > 1$ in A (by the choice of order of A).

So, $\bar{a}^{\tau(\chi)} > 1$ (since b, s, χ_s and $n_1, n_2, \ldots, n_s$ always satisfy one of the conditions of (3.1.6)). //

We are now able to give the required embedding theorem.

THEOREM 1. *Let* G *be an MO-group with* $A = I(G')$ *and* $\Phi = G/A$. *Then* G *can be embedded in an MO-group,* $\bar{G}$ *, which is a split extension of an abelian group,* $\bar{A}$ *, by* Φ .

Proof. It remains to show that $\bar{G}$ is an O-group. If we order $\bar{A}$ as in the previous lemma and if we choose any order of Φ , then it is easy to show that the following definition makes $\bar{G}$ an o-group:-

For ϕ in Φ and $\bar{a}$ in $\bar{A}$, define

$(\phi, \bar{a}) > 1$ if, and only if, $\phi > 1$, *or* $\phi = 1$ and $\bar{a} > 1$. //

It is worth noting that at no stage during the construction or ordering of $\bar{G}$ did we *rely* on the fact that Φ is abelian. Theorem 1 can be generalized as follows:-

THEOREM 1'. *Let* G *be an O-group with normal, abelian, relatively convex subgroup,* A . *Then* G *can be embedded in an O-group,* $\bar{G}$ *, which is a split extension of an abelian group,* $\bar{A}$ *, by* G/A .

By Theorem 1, step (1) of our "possible approach" (see Section 3.0) has been accomplished. Setting $A_1 = \bar{A}$ and $G_1 = \bar{G}$, we proceed to the next step.

3.2 Completing the derived group

Strictly speaking, we complete A_1 (but in the process, of course, G_1' is completed). Let $A_2 = A_1^{\#}$ and $G_2 = G_1^{\#}$ (as in Section 2.1). Then G_2 splits over A_2 (by 2.3 Theorem 12) and G_2 is an MO-group (*cf.* 2.3 Theorem 11). (Furthermore, the embedding of G_1 into G_2 is an o^*-embedding (by 2.2 Theorem 3).)

3.3 The endomorphisms $1 + \phi + \phi^2 + \ldots + \phi^{n-1}$

3.3.1. We commence this section with two changes of notation which will be used throughout this section and the next chapter.

Given an MO-group, G, we shall suppose that G splits over A, where A is a divisible, normal, abelian subgroup of G and G/A is abelian. So there is a homomorphism from G/A into $\mathrm{aut}(A)$ which defines the extension of A by G/A (see Kurosh [17], p. 149). The first change we make is to denote by Φ the homomorphic image of G/A in $\mathrm{aut}(A)$. Hence, for the remainder of this thesis, we shall be discussing semi-direct products of abelian O-groups by abelian O-automorphism groups. 0.3 Theorem 6 will provide the link between these and the MO-groups under consideration.

The second, and less subtle, change is to write A *additively*. The reason for this is that we shall sometimes regard A as a vector space (over $\mathbb{Q}$). This means that multiplication in the semi-direct product of A by an automorphism group, Φ, will be given by

$$(\phi, a)(\chi, b) = (\phi\chi, a\chi+b) \qquad (3.3.1)$$

for all ϕ, χ in Φ and a, b in A. The reason we do not make the change complete by writing everything additively, is that we use the natural ring structure of $\mathrm{aut}(A)$ (see, for example, Fuchs [6], p. 210 ff).

3.3.2. Suppose that A is a divisible, abelian O-group (that is, torsion-free group - see Fuchs [7], p. 36) with Φ an abelian, O-automorphism group of A. Then $F = A\lambda\Phi$ is an O-group (0.3 Theorem 6). Suppose, furthermore, that Φ is divisible. (Here we are saying, in effect - suppose that step (3) of our "possible approach" has been accomplished.) What else is required to make F divisible?

Since $(\phi, a)^n = \left(\phi^n, a + a\phi + \ldots + a\phi^{n-1}\right)$ for all (ϕ, a) in F and n in $\mathbb{Z}^+$, we have, for given (χ, b) in F, $(\phi, a)^n = (\chi, b)$ if, and only if, $\phi^n = \chi$ and $a + a\phi + \ldots + a\phi^{n-1} = b$. Φ is divisible, so there is such a ϕ in Φ. This means that F will be divisible if, and only if, for all ϕ in Φ, a in A and n in $\mathbb{Z}^+$, the equation $x + x\phi + \ldots + x\phi^{n-1} = a$ has a solution for x in A, or, equivalently, each endomorphism $1 + \phi + \ldots + \phi^{n-1}$ is onto. That this is not generally the case (even when Φ is divisible) is shown by the following example.

Let A be the vector space (over $\mathbb{Q}$) with basis $\{a_i : i \in \mathbb{Q}\}$ and let Φ be $\{\phi^\alpha : \alpha \in \mathbb{Q}\}$. (That is, $\phi^\alpha\phi^\beta = \phi^{\alpha+\beta}$, so Φ is (isomorphic to) $\mathbb{Q}$.) The mapping $a_i\phi^\alpha = a_{i+\alpha}$ of the basis of A onto itself induces an automorphism of A which is an o-automorphsm if A is ordered by $\sum_{i \in \mathbb{Q}} r_i a_i > 0$ if, and only if, $r_j > 0$ in $\mathbb{Q}$ where $i > j$ in $\mathbb{Q}$ implies $r_i = 0$. (Remember that the sum is finite.) Furthermore, $\left(a_i\phi^\alpha\right)\phi^\beta = a_i\left(\phi^\alpha\phi^\beta\right)$ for all i, α, β in $\mathbb{Q}$; so, the mapping $a \mapsto a\phi^\alpha$ (a in A, α in $\mathbb{Q}$) is an o-action of Φ on A, whence, the semi-direct product, $A\lambda\Phi$, is an O-group (by 0.3 Lemma 1). Since, for all $a = \sum_i r_i a_i$ in A, $a + a\phi^1 = \sum_i r_i a_i + \sum_i r_i a_{i+1}$, the equation $x + x\phi^1 = a_0$ has no solution for x in A.

3.3.3. Now to the problem of making the endomorphisms $1 + \phi + \ldots + \phi^{n-1}$ onto for all ϕ in Φ. We proceed as follows:-

THEOREM 2. *Let A be a divisible abelian O-group with Φ an abelian O-automorphism group of A. Suppose there are a_0 in A and ψ in Φ such that the equation $x + x\psi = a_0$ has no solution for x in A. Then A can be embedded in a divisible, abelian O-group, $\tilde{A}$, satisfying:-*

(i) each ϕ in Φ extends to $\tilde{\phi}$ in $\mathrm{aut}(\tilde{A})$,

(ii) the mapping $\phi \mapsto \tilde{\phi}$ is an isomorphism from Φ into $\mathrm{aut}(\tilde{A})$

(and we write $\tilde{\Phi} = \{\tilde{\phi} : \phi \in \Phi\}$ *),*

(iii) there is x *in* $\tilde{A}$ *such that* $x + x\tilde{\psi} = a_0$ *, and*

(iv) $\tilde{\Phi}$ *is an* O*-automorphism group of* $\tilde{A}$.

The proof follows shortly. After a certain amount of transfinite and induction argument, we have:-

THEOREM 3. *Let* A *be a divisible, abelian* O*-group with* Φ *an abelian* O*-automorphism group of* A *. Then* A *can be embedded in a divisible, abelian* O*-group,* $\ddot{A}$ *, satisfying:-*

(i) each ϕ *in* Φ *extends to* $\ddot{\phi}$ *in* $\mathrm{aut}(\ddot{A})$ *,*

(ii) the mapping $\phi \mapsto \ddot{\phi}$ *is an isomorphism from* Φ *into* $\mathrm{aut}(\ddot{A})$ *,*

(iii) for all $\phi_1, \phi_2, \ldots, \phi_n$ *in* Φ *, the endomorphism* $\ddot{\phi}_1 + \ddot{\phi}_2 + \ldots + \ddot{\phi}_n$ *is onto, and*

(iv) $\ddot{\Phi}$ *is an* O*-automorphism group of* $\ddot{A}$ *(where* $\ddot{\Phi}$ *has the obvious meaning).*

The proof of this theorem, requiring knowledge of the construction of $\tilde{A}$ in the previous theorem, is postponed. As an almost immediate corollary to Theorem 3, we have:-

THEOREM 4. *Let* G *be an* MO*-group with normal, abelian, isolated subgroup,* A *, such that* $I(G') \leq A$ *. If* G/A *is divisible, then* G *can be embedded in a divisible* MO*-group.*

Proof. Suppose that G/A is divisible. First (by 3.1 Theorem 1), we embed G in $\bar{G}$, an MO-group which is a split extension of $\bar{A}$ by G/A , where $\bar{A}$ is an abelian O-group. Next embed $\bar{G}$ in $H = \bar{G}^{\#}$, an MO-group which is a split extension of $B = \bar{A}^{\#}$ by G/A , where B is a divisible, abelian O-group (2.3 Theorems 11 and 12). Let $\varphi : G/A \to \mathrm{aut}(B)$ be the homomorphism corresponding to this split extension, and write $\Phi = (G/A)\varphi$. Then Φ is a (divisible, abelian) O-automorphism group of B (by 0.3 Theorem 6). Let $B \to \ddot{B}$ be the embedding of Theorem 3 of this section and let $\ddot{H}$ be the split extension (equivalently, semi-direct product) of $\ddot{B}$ by G/A, where the homomorphism, $\ddot{\varphi}$, from G/A into $\mathrm{aut}(\ddot{B})$ is the product of the given homomorphisms $\varphi : G/A \to \Phi$ and $\phi \longmapsto \ddot{\phi}$ (ϕ in Φ). So, $\ddot{H}$ is metabelian. By Theorem 3 *(iv)*, $\ddot{B}\lambda\ddot{\Phi}$ is an O-group; whence, $\ddot{\Phi}$ is an O-automorphism group of $\ddot{B}$ (0.3 Theorem 6) and so, $\ddot{H}$ is an O-group (0.3 Theorem 6 again). Since the embedding, B into $\ddot{B}$, induces a natural embedding of H into $\ddot{H}$, the theorem follows once we have shown that $\ddot{H}$ is divisible.

Writing the coset gA as g' for all g in G, we take any (g_0', b_0) in $\ddot{H}$ (that is g_0' is in G/A and b_0 is in $\ddot{B}$) and any n in Z^+. The action of G/A on $\ddot{B}$ is given by $bg' = b(g'\ddot{\varphi})$ for all b in $\ddot{B}$ and g' in G/A. (Since $\ddot{B}$ is additive, we write bg' rather than $b^{g'}$.) So, there is (g', b) in $\ddot{H}$ such that $(g', b)^n = (g_0', b_0)$ if, and only if, $(g')^n = g_0'$ in G/A and $b\left(1 + g'\ddot{\varphi} + \ldots + (g'\ddot{\varphi})^{n-1}\right) = b_0$. G/A is divisible, so such a g' exists; and by Theorem 3, there is such a b in $\ddot{B}$. //

3.3.4. We turn now to the task of proving Theorems 2 and 3, and commence with some definitions and lemmas.

For an arbitrary group, G, with subgroup, F, and endomorphism, γ, we say that F is *γ-invariant* if $F\gamma \leq F$. For a subgroup, Γ, of $\mathrm{aut}(G)$, we say that F is *Γ-characteristic* if F is γ-invariant for all γ in Γ. Given that F is γ-invariant, we denote by $\gamma|_F$ the restriction of γ to F.

LEMMA 3. *Let W be an abelian group. Suppose $W = \mathrm{gp}(U, V_0)$ where U, V_0 are subgroups of W and U is divisible. Then $W = U + V$ (a direct sum) for some subgroup, V, of V_0.*

Proof. Take $V \leq V_0$ such that $V \cap U = \{0\}$ and V is maximal with respect to this property. The proof in Fuchs [6], p. 63, commencing

"Let H be an arbitrary subgroup ..." can be transferred directly to this case. //

LEMMA 4. *Let W be an abelian O-group and let $\omega_1, \omega_2, \ldots, \omega_n$ belong to some O-automorphism group of W. Then the endomorphism $\omega_1 + \omega_2 + \ldots + \omega_n$ is one-to-one and order preserving for some order of W.*

Proof. Let $\leq$ be an order of W under which each ω_j $(j = 1, 2, \ldots, n)$ is an o-automorphism. Then, for all $w > 0$ in W and $j = 1, 2, \ldots, n$, $w\omega_j > 0$. So, $w > 0$ implies $w(\omega_1 + \omega_2 + \ldots + \omega_n) = w\omega_1 + w\omega_2 + \ldots + w\omega_n > 0$. Similarly, $w < 0$ implies $w(\omega_1 + \omega_2 + \ldots + \omega_n) < 0$. So, $\omega_1 + \omega_2 + \ldots + \omega_n$ is both order preserving and one-to-one. //

(We remind the reader that abelian groups are being written additively unless they are automorphism groups, in which case, the natural ring structure of the (full) automorphism group of an abelian group will be used.)

PROOF OF THEOREM 2. First we construct an extension, $\tilde{A}$, of A; then, by way of Lemmas 6, 7, 8 and 11 respectively, show that $\tilde{A}$ satisfies *(i)*, *(ii)*, *(iii)* and *(iv)* of Theorem 2.

Suppose A, Φ, a_0 and ψ satisfy the hypotheses of Theorem 2. To construct $\tilde{A}$, we are going to take a divisible, abelian O-group, D, which is isomorphic to a particular subgroup, C, of A, and set $\tilde{A} = A + D$. (We mention once and for all that all sums of groups in this section are *direct*.)

Let $B = \{a+a\psi : a \in A\}$. We have:-

LEMMA 5. *B is a divisible subgroup of A and is Φ-characteristic.*

Proof. For all a_1, a_2 in A,

$$(a_1+a_1\psi) - (a_2+a_2\psi) = (a_1-a_2) + (a_1-a_2)\psi ;$$

so, B is a subgroup.

For all a in A and m in Z^+,

$$\begin{aligned} m\{(1/m)a+\{(1/m)a\}\psi\} &= m\{(1/m)a+(1/m)(a\psi)\} \\ &= m(1/m)(a+a\psi) \\ &= a + a\psi \end{aligned}$$

where $(1/m)a$ is the element, y, of A satisfying $my = a$. Since A is divisible, such y in A always exists, and it follows that B is divisible.

For all a in A and ϕ in Φ,

$$(a+a\psi)\phi = a\phi + (a\psi)\phi = a\phi + (a\phi)\psi \quad \text{(since } \Phi \text{ is abelian).}$$

So, B is Φ-characteristic. //

It is convenient to regard A as a vector space over the rationals (see Fuchs [6], p. 64). For an arbitrary vector space, W, we shall denote by $VS(X)$ the vector subspace of W generated by a subset, X, of W, and we shall treat this notation as freely as the $\mathrm{gp}(X)$ notation (see Section 0.2, p. 6).

Let $C_0 = VS\{a_0\phi : \phi \in \Phi\}$ and take $C \le C_0$ maximal with respect to intersecting B in $\{0\}$. So, by Lemma 3 of this section, $VS\{B, C_0\} = B + C$. We shall write arbitrary elements of C_0 in the form

$$a_0\Delta = \sum_{\phi\in\Phi} \alpha_\phi a_0\phi \quad \text{where} \quad \alpha_\phi \in Q .$$

(The sum is finite.) So, each $\Delta = \sum_{\phi\in\Phi} \alpha_\phi\phi$ is simply an endomorphism of A.

Let $\{a_0\Delta_i : i \in I\}$ be a basis for C. Since $VS(a_0) \cap B = \{0\}$ (otherwise $x + x\psi = a_0$ has a solution for x in A), then C can be chosen containing $VS(a_0)$ (see Fuchs [6], p. 63); so we may, and shall, suppose that a_0 is in the basis of C. That is, $a_0\Delta_i = a_0$ for some i in I - we shall suppose that this Δ_i is 1. D is the vector space (over Q) with basis $\{x\Delta_i : i \in I\}$, where, for the moment, $x\Delta_i$ is simply a new symbol. Let $\iota : D \to C$ be the isomorphism

$$\iota\left(\sum_i \delta_i x\Delta_i\right) = \sum_i \delta_i a_0 \Delta_i$$

(where, of course, i ranges over I and the sums are finite).

Setting $\tilde{A} = A + D$, we define for ϕ in Φ and $\tilde{a} = a + d$ in $\tilde{A}$, $\tilde{a}\tilde{\phi} = a' + d'$, where a' in A and d' in D are given by

$$\big(a+a\psi+\iota(d)\big)\phi = a' + a'\psi + \iota(d') . \qquad (3.3.2)$$

Observe that B (by Lemma 5) and C_0 (obvious from its definition) are Φ-characteristic, and so $B + C = VS(B, C_0)$ is Φ-characteristic. Furthermore, the fact that the sum $B + C$ is direct ensures that, given a in A and d in D , the $a' + a'\psi$ in B and $\iota(d')$ in C are unique. Since $a' + a'\psi = a'(1+\psi)$, where $1 + \psi$ is one-to-one (by Lemma 4), and since ι is an isomorphism, it follows that the a' in A and d' in D are unique. So, (3.3.2) is a sensible definition.

To prove *(i)* of Theorem 2, we have:-

LEMMA 6. *For all* ϕ *in* Φ , $\tilde{\phi}$ *is an automorphism of* $\tilde{A}$ *and* $\tilde{\phi}|_A = \phi$.

Proof. Take any $\tilde{a}_1 = a_1 + d_1$ and $\tilde{a}_2 = a_2 + d_2$ in $\tilde{A}$ and ϕ in Φ . Then

$$(\tilde{a}_1+\tilde{a}_2)\tilde{\phi} = a' + d' \quad\text{and}\quad \tilde{a}_1\tilde{\phi} + \tilde{a}_2\tilde{\phi} = (a_1'+a_2') + (d_1'+d_2') ,$$

where

$$\big(a_1+a_2+(a_1+a_2)\psi+\iota(d_1+d_2)\big)\phi = a' + a'\psi + \iota(d') , \qquad (3.3.3)$$

$$\big(a_1+a_1\psi+\iota(d_1)\big)\phi = a_1' + a_1'\psi + \iota(d_1') \qquad (3.3.4)$$

and

$$\big(a_2+a_2\psi+\iota(d_2)\big)\phi = a_2' + a_2'\psi + \iota(d_2') . \qquad (3.3.5)$$

Observing that the left hand sides of (3.3.4) and (3.3.5) add up to that of (3.3.3), the same is true for the corresponding right hand sides.

$B + C$ is direct, so

$$a' + a'\psi - (a_1'+a_1'\psi) - (a_2'+a_2'\psi) = 0 \quad \text{and} \quad \iota(d') - \iota(d_1') - \iota(d_2') = 0 .$$

That is,

$$(a'-(a_1'+a_2'))(1+\psi) = 0 \quad \text{and} \quad \iota(d'-(d_1'+d_2')) = 0 .$$

By Lemma 4, $a' - (a_1'+a_2') = 0$ and, since ι is an isomorphism, $d' - (d_1'+d_2') = 0$. So, $(\tilde{a}_1+\tilde{a}_2)\tilde{\phi} = \tilde{a}_1\tilde{\phi} + \tilde{a}_2\tilde{\phi}$.

To show that $\tilde{\phi}|_A = \phi$, take any a in A . Then, by (3.3.2), $(a+0)\tilde{\phi} = a' + d'$ where $(a+a\psi+\iota(0))\phi = a' + a'\psi + \iota(d')$. Since B is ϕ-invariant (Lemma 5), it follows that $\iota(d')$, and hence d' , is 0 . So $a\tilde{\phi} = a'$ and $a\phi + (a\phi)\psi = (a+a\phi)\psi = a' + a'\psi$; whence $a\tilde{\phi} = a\phi$.

Now take $\tilde{a} = a + d \neq 0$ in $\tilde{A}$. So $\tilde{a}\tilde{\phi} = a\phi + d\tilde{\phi}$. If $d \neq 0$, then $d\tilde{\phi}$ is not in A (otherwise, by (3.3.2), $(\iota(d))\phi = a' + a'\psi$ for some $a' \neq 0$ in A ; that is, $(\iota(d))\phi$ is in B . Whence, $\iota(d) = (\iota(d))\phi\phi^{-1}$ is in B (Lemma 5), which is impossible for $d \neq 0$ since $B \cap C = \{0\}$). So, $d \neq 0$ implies $\tilde{a}\tilde{\phi} \neq 0$. If $d = 0$, then $a \neq 0$ and $\tilde{a}\tilde{\phi} = a\phi \neq 0$.

Finally we show that $\tilde{\phi}$ is onto. Take any $\tilde{a}_1 = a_1 + d_1$ in $\tilde{A}$. Since ϕ is in $\mathrm{aut}(A)$ and since $B + C$ is ϕ-invariant, there exist a in A and $\iota(d)$ in C such that $(a+a\psi+\iota(d))\phi = a_1 + a_1\psi + \iota(d_1)$. By (3.3.2), $(a+d)\tilde{\phi} = \tilde{a}_1$. //

The next lemma proves *(ii)* of Theorem 2.

LEMMA 7. *The mapping* $\phi \mapsto \tilde{\phi}$ *is an isomorphism from* Φ *into* $\mathrm{aut}(\tilde{A})$.

Proof. By the previous lemma, the range of the mapping is in $\mathrm{aut}(\tilde{A})$. The mapping is one-to-one because $\tilde{\phi}|_A = \phi$ implies that if $\phi \neq 1$, then $\tilde{\phi} \neq 1$. It remains to show that $(\widetilde{\phi\chi}) = \tilde{\phi}\tilde{\chi}$ for all ϕ, χ in Φ . Take any $\tilde{a} = a + d$ in $\tilde{A}$. Then, $\tilde{a}\tilde{\phi} = a' + d'$, $(\tilde{a}\tilde{\phi})\tilde{\chi} = \tilde{a}(\tilde{\phi}\tilde{\chi}) = a'' + d''$, and $\tilde{a}(\widetilde{\phi\chi}) = a''' + d'''$, where

$$\{a+a\psi+\iota(d)\}\phi = a' + a'\psi + \iota(d') ,$$

$$\{a'+a'\psi+\iota(d')\}\chi = a'' + a''\psi + \iota(d'') ,$$

and

$$\{a+a\psi+\iota(d)\}(\phi\chi) = a''' + a'''\psi + \iota(d''') .$$

Since $(\{a+a\psi+\iota(d)\}\phi)\chi = \{a+a\psi+\iota(d)\}(\phi\chi)$, it follows that $a'' + a''\psi + \iota(d'') = a''' + a'''\psi + \iota(d''')$; whence $a'' = a'''$ and $d'' = d'''$. That is, $\tilde{a}(\widetilde{\phi\chi}) = \tilde{a}(\tilde{\phi}\tilde{\chi})$. //

Lemmas 6 and 7 imply that Φ is (isomorphic to) an automorphism group of $\tilde{A}$. Accordingly, we shall not retain the $\tilde{\phi}$ notation for ϕ in Φ , writing merely ϕ whether the domain be A or $\tilde{A}$.

The element in $\tilde{A}$ satisfying *(iii)* of Theorem 2 is $0 + x1$, which we shall denote by x . (Remember that the basis, $\{a_0\Delta_i : i \in I\}$, of C has been chosen so that one of the endomorphisms, Δ_i , is 1 .) Our construction of $\tilde{A}$ makes the next lemma almost obvious.

LEMMA 8. $x + x\psi = a_0$.

Proof. $a_0\psi = a_0\psi + (a_0-a_0) = (a_0+a_0\psi) - a_0$. Since $a_0 = \iota(x)$, we have $\{0+0\psi+\iota(x)\}\psi = a_0 + a_0\psi + \iota(-x)$. So, by (3.3.2), $x\psi = (0+x)\psi = a_0 - x$; whence $x + x\psi = a_0$. //

Before we can prove *(iv)* of Theorem 2, we need the following lemma and important corollary.

LEMMA 9. *Let* $\Delta = \alpha_1\phi_1 + \alpha_2\phi_2 + \ldots + \alpha_k\phi_k$ (α_j *in* Q , ϕ_j *in* Φ *for* $j = 1, 2, \ldots, k$) *be one of the* Δ_i (i *in* I) . (*That is,* $a_0\Delta$ *is in the chosen basis of* C .) *Then* $x\Delta = \alpha_1x\phi_1 + \alpha_2x\phi_2 + \ldots + \alpha_kx\phi_k$.

Proof. Setting $a_0\phi_j = a_j + a_j\psi + \iota(d_j)$, where a_j is in A and d_j in D for $j = 1, 2, \ldots, k$, we have $x\phi_j = a_j + d_j$ for all $j = 1, 2, \ldots, k$ and

$$a_0\Delta = \sum_{j=1}^{k} \alpha_j a_0 \phi_j$$

$$= \sum_{j=1}^{k} \alpha_j \left(a_j + a_j\psi + \iota\left(d_j\right)\right)$$

$$= \sum_{j=1}^{k} \alpha_j a_j + \left(\sum_{j=1}^{k} \alpha_j a_j\right)\psi + \iota\left(\sum_{j=1}^{k} \alpha_j d_j\right) .$$

Now $a_0\Delta$ and $\iota\left(\sum_{j=1}^{k} \alpha_j d_j\right)$ are in C, so $\sum_{j=1}^{k} \alpha_j a_j + \left(\sum_{j=1}^{k} \alpha_j a_j\right)\psi$ is in C and, hence, in $B \cap C = \{0\}$. So, by Lemma 4, $\sum_{j=1}^{k} \alpha_j a_j = 0$. Hence, we have

$$\sum_{j=1}^{k} \alpha_j x\phi_j = \sum_{j=1}^{k} \alpha_j\left(a_j + d_j\right)$$

$$= \sum_{j=1}^{k} \alpha_j a_j + \sum_{j=1}^{k} \alpha_j d_j$$

$$= 0 + \iota^{-1}\left(a_0\Delta\right)$$

$$= x\Delta \quad \text{(by definition of } \iota \text{)}. \qquad //$$

COROLLARY 1. *For all* $\tilde{a}$ *in* $\tilde{A}$, $\tilde{a} + \tilde{a}\psi$ *is in* A.

Proof. Take $\tilde{a} = a + d$ in $\tilde{A}$ where $d = \sum_i \delta_i x\Delta_i$. Now, $\tilde{a} + \tilde{a}\psi = (a+a\psi) + (d+d\psi)$, so it is necessary to show that $d + d\psi$ is in A. Observing that Δ_i (for all i in I) is an endomorphism of $\tilde{A}$ which commutes with each ϕ in Φ, we have:-

$$d + d\psi = \sum_i \delta_i x\Delta_i + \left(\sum_i \delta_i x\Delta_i\right)\psi$$

$$= \sum_i \delta_i \left(x\Delta_i + \left(x\Delta_i\right)\psi\right)$$

$$= \sum_i \delta_i \left(x\Delta_i + (x\psi)\Delta_i\right)$$

$$= \sum_i \delta_i (x + x\psi)\Delta_i$$

$$= \sum_i \delta_i a_0 \Delta_i$$

$$= \iota(d) .$$

So, $d + d\psi$ is in C and, hence, $\tilde{a} + \tilde{a}\psi$ is in A. //

This corollary suggests a way of ordering $\tilde{A}$ so that each ϕ in Φ is an o-automorphism of $\tilde{A}$ - namely:-

$$\left.\begin{array}{l}\text{Take an order, } \leq \text{, of } A \text{ with respect to which } \Phi \text{ is an} \\ o\text{-automorphism group.} \\ \text{For } \tilde{a} \text{ in } \tilde{A} \text{, define } \tilde{a} > 0 \text{ if, and only if, } \tilde{a} + \tilde{a}\psi > 0 \\ \text{in } A\text{.}\end{array}\right\} \quad (3.3.6)$$

LEMMA 10. *$(\tilde{A}, \leq)$ is an o-group.*

Proof. From the proof of Corollary 1, we have $\tilde{a} + \tilde{a}\psi = a + a\psi + \iota(d)$ for all $\tilde{a} = a + d$ in $\tilde{A}$. So, (3.3.6) may be rewritten

$$\tilde{a} = a + d > 0 \text{ if, and only if, } a + a\psi + \iota(d) > 0 \text{ in } A\,. \quad (3.3.7)$$

Hence, for all $\tilde{a} = a + d$ in $\tilde{A}$,

$$-\tilde{a} > 0 \text{ if, and only if, } -\left(a+a\psi+\iota(d)\right) > 0 \text{ in } A\,;$$

whence, $\tilde{a} > 0$ implies $-\tilde{a} \ngtr 0$, and for $\tilde{a} \neq 0$, either $\tilde{a} > 0$ or $-\tilde{a} > 0$. (Observe that if $a + a\psi + \iota(d) = 0$, then $a + a\psi = 0$ and $\iota(d) = 0$ (since $B + C$ is direct), and so $a = 0$ and $d = 0$.)

Now take $\tilde{a}_1 = a_1 + d_1 > 0$ and $\tilde{a}_2 = a_2 + d_2 > 0$ in $\tilde{A}$. That is (by (3.3.7)), $a_1 + a_1\psi + \iota(d_1) > 0$ and $a_2 + a_2\psi + \iota(d_2) > 0$ in A, whence $(a_1+a_2) + (a_1+a_2)\psi + \iota(d_1+d_2) > 0$ in A. Again by (3.3.7), $\tilde{a}_1 + \tilde{a}_2 > 0$. Since $\tilde{A}$ is abelian, 0.2 Theorem 1 shows that $(\tilde{A}, \leq)$ is an o-group. //

LEMMA 11. *Φ is an O-automorphism group of $\tilde{A}$.*

Proof. Order $\tilde{A}$ by (3.3.6) and take $\tilde{a} = a + d > 0$ in $\tilde{A}$ and any ϕ in Φ. By (3.3.2), $\tilde{a}\phi = a' + d'$ where

$$\left(a+a\psi+\iota(d)\right)\phi = a' + a'\psi + \iota(d')\,.$$

Since $a + a\psi + \iota(d) > 0$ in A, it follows that $\left(a+a\psi+\iota(d)\right)\phi > 0$ in A (by (3.3.6)); whence $\tilde{a}\phi > 0$. By the arbitrary choice of ϕ in Φ, the required result follows. //

So, Lemmas 6, 7, 8 and 11 combine to prove Theorem 2.

PROOF OF THEOREM 3. We prove Theorem 3 by induction on n and commence with the case $n = 2$.

Let A be a divisible, abelian O-group with Φ an abelian O-automorphism group of A. Taking any ϕ_1, ϕ_2 in Φ, we show that A can be embedded in a divisible, abelian O-group, $\ddot{A}$, satisfying *(i)*, *(ii)*, *(iii)* and *(iv)* of Theorem 3.

Since $\phi_1 + \phi_2 = \left(1+\phi_2\phi_1^{-1}\right)\phi_1$ in the endomorphism ring of A, and since ϕ_1 and $\phi_2\phi_1^{-1}$ are in Φ, it suffices to show that $1 + \phi_2\phi_1^{-1}$ can be made onto. Theorem 2 (with $\psi = \phi_2\phi_1^{-1}$) and the usual transfinite arguments show that such an $\ddot{A}$ exists.

The induction hypothesis is the following:-

Let A and Φ be as for $n = 2$ case. Take $n \geq 2$ and suppose that for all $\phi_1, \phi_2, \ldots, \phi_{n-1}$ in Φ, the endomorphism $\phi_1 + \phi_2 + \ldots + \phi_{n-1}$ is onto.

Take any $\phi_1, \phi_2, \ldots, \phi_n$ in Φ and set $\chi = \phi_1 + \phi_2 + \ldots + \phi_{n-1}$. By the induction hypothesis, χ is onto, and by Lemma 4 of this section, χ is an O-automorphism of A. In fact, $\Phi_1 = \text{gp}(\Phi, \chi)$ is an O-automorphism group of A. Since Φ is abelian, χ centralizes Φ, and so Φ_1 is abelian.

Suppose a_0 is in A such that $x(\chi+\phi_n) = a_0$ has no solution for x in A. (If there is no such a_0 in A, that is, if $\chi + \phi_n$ is onto, then we have nothing to prove.) Applying Theorem 2 to A, Φ_1, $a_0\chi^{-1}$ and $\phi_n\chi^{-1}$, we obtain $\tilde{A}$ with an element, x, such that $x\left(1+\widetilde{\left(\phi_n\chi^{-1}\right)}\right) = a_0\chi^{-1}$; so,

$$x(\tilde{\chi}+\tilde{\phi}_n) = x\{1+(\tilde{\phi}_n\tilde{\chi}^{-1})\}\tilde{\chi}$$

$$= x\{1+(\widetilde{\phi_n\chi^{-1}})\}\tilde{\chi} \quad (\phi \mapsto \tilde{\phi} \ (\phi \text{ in } \Phi_1) \text{ is a homomorphism})$$

$$= \{a_0\chi^{-1}\}\chi \quad (\tilde{\chi}|_A = \chi)$$

$$= a_0 .$$

It remains to show that $\tilde{\chi} = \tilde{\phi}_1 + \tilde{\phi}_2 + \ldots + \tilde{\phi}_{n-1}$. From this, it follows that $x(\tilde{\phi}_1 + \tilde{\phi}_2 + \ldots + \tilde{\phi}_n) = a_0$. Once more applying the usual transfinite arguments, we can "make" $\phi_1 + \phi_2 + \ldots + \phi_n$ onto a suitable extension, $\ddot{A}$, of A in such a way that the rest of the theorem (*(i)*, *(ii)* and *(iv)*) follows as well.

To show that $\tilde{\chi} = \tilde{\phi}_1 + \tilde{\phi}_2 + \ldots + \tilde{\phi}_{n-1}$, we set $\psi = \phi_n\chi^{-1}$ in Φ_1 and take any $\tilde{a} = a + d$ in $\tilde{A}$. (Remember that $\tilde{A} = A + D$ for a suitable divisible, abelian O-group, D. See proof of Theorem 2.)

Now $\tilde{a}\tilde{\chi} = a' + d'$, and $\tilde{a}\tilde{\phi}_j = a'_j + d'_j$ for all $j = 1, 2, \ldots, n-1$, where

$$(a+a\psi+\iota(d))\chi = a' + a'\psi + \iota(d') ,$$

and

$$(a+a\psi+\iota(d))\phi_j = a'_j + a'_j\psi + \iota(d'_j) \quad \text{for all } j = 1, 2, \ldots, n-1 .$$

Since $(a+a\psi+\iota(d))\chi = (a+a\psi+\iota(d))(\phi_1 + \phi_2 + \ldots + \phi_{n-1})$, it follows that $a' = a'_1 + a'_2 + \ldots + a'_{n-1}$ and $d' = d'_1 + d'_2 + \ldots + d'_{n-1}$; whence $\tilde{a}\tilde{\chi} = \tilde{a}(\tilde{\phi}_1 + \tilde{\phi}_2 + \ldots + \tilde{\phi}_{n-1})$. //

CHAPTER 4

A PROPOSITION: SOME CLUES AND AN EXAMPLE

4.0 Introduction

As was mentioned in Chapter 3, Kokorin's question ([14], question 1.60) still lacks a complete answer. The purpose of this chapter is to discuss the third step of our "possible approach" (see Section 3.0) and, hopefully, to shed some light on the problem.

In 4.1 we give a Proposition which, in Theorem 1, is shown to be equivalent to the statement:-

Every *MO*-group can be embedded in a divisible *MO*-group.

In 4.2 and 4.3, we consider a special case of the Proposition, and in 4.4 we present an example which not only illustrates some of the difficulties of the problem but also demonstrates that if Kokorin's question can be answered affirmatively, then a metabelian, orderable *d*-closure of an *MO*-group need not be unique.

4.1 The proposition

The proposition, which we present below and hereafter denote by P , says, in effect, that we can adjoin roots to an abelian *O*-automorphism group of an abelian *O*-group in such a way that the resulting group is sill an abelian *O*-automorphism group of a (possibly larger) abelian *O*-group. In view of 3.3 Theorem 4, we are able to show (Theorem 1, below) that P is equivalent to the statement:-

<u>M</u> *An MO-group can be embedded in a divisible MO-group.*

The proposition, P , is the following:-

<u>P</u> *Let* A *be a divisible, abelian O-group with* Φ *an abelian O-automorphism group of* A *. Then* A *can be extended to a divisible,*

abelian O-group, $\hat{A}$ *, such that* $\Phi^{\#}$ *, the abelian d-closure of* Φ *, is an O-automorphism group of* $\hat{A}$ *.*

THEOREM 1. $P \Leftrightarrow M$.

Proof. Suppose P is true and let G be any MO-group. By 3.1 Theorem 1 and 2.3 Theorems 11 and 12, we suppose, without loss of generality, that G splits over A , where A is a normal, abelian, divisible subgroup of G and $I(G') \leq A$. Let φ and Φ be as usual (see the remarks at the beginning of Section 3.3). Let $\hat{A}$ be the extension of A given by P . Writing $(G/A)^{\#}$ for the abelian d-closure of G/A , we have (by 0.2 Theorem 4) that the epimorphism, $\varphi : G/A \to \Phi$, extends (uniquely) to a homomorphism, $\varphi^{\#} : (G/A)^{\#} \to \Phi^{\#}$. $\varphi^{\#}$ is an epimorphism because Φ is a subgroup of $(G/A)^{\#}\varphi^{\#}$ which is divisible. So, $\varphi^{\#}$ is a homomorphism from $(G/A)^{\#}$ into $\mathrm{aut}(\hat{A})$ and we write $\hat{G}$ for the split extension of $\hat{A}$ by $(G/A)^{\#}$ corresponding to this homomorphism. Clearly $\hat{G}$ is metabelian, and by P and 0.3 Theorem 6, $\hat{G}$ is an O-group. Furthermore, G embeds naturally into $\hat{G}$.

Conversely, suppose that M is true. Let A and Φ satisfy the hypotheses of P and write $G = A\lambda\Phi$ (the action of Φ on A being the obvious one). By M , there is a d-extension, H , of G such that H is an MO-group.

[For the purposes of this proof, we return to purely multiplicative notation and regard G as being generated by its subgroups A and Φ where A is normal and $A \cap \Phi = \{1\}$. (See 0.3 Theorem 5.) This means that for f in Φ and a in A , the element $f^{-1}af$ in A is simply the image of a under the automorphism, f .]

Let $\Phi^{\#}$ be the isolated closure of Φ in H . Then $\Phi^{\#}$ is abelian (by 2.1 Lemma 2) and is divisible (since H is divisible). Setting $\hat{A} = A^{\Phi^{\#}}$ (the group generated by all elements $f^{-1}af$, where f and a

range over $\Phi^{\#}$ and A respectively), we show that $\hat{A}$ is a normal, divisible, abelian subgroup of $\hat{G} = \mathrm{gp}(\hat{A}, \Phi^{\#})$ and that $\hat{A} \cap \Phi^{\#} = \{1\}$. It follows from this that $\hat{G}$ is a semi-direct product of $\hat{A}$ by $\Phi^{\#}$ (0.3 Theorem 5), whence $\Phi^{\#}$ has a homomorphic image which is an automorphism group of $\hat{A}$. To show that $\Phi^{\#}$ itself is (isomorphic to) an automorphism group of $\hat{A}$, take $f_1 \neq f_2$ in $\Phi^{\#}$. Take m in Z^+ such that f_1^m and f_2^m are in Φ. Since $\Phi^{\#}$ is an O-group, $f_1^m \neq f_2^m$ and so $\left(f_1^m\right)^{-1} a f_1^m \neq \left(f_2^m\right)^{-1} a f_2^m$ for some a in A; that is, $\left[a, \left(f_1 f_2^{-1}\right)^m\right] \neq 1$ in G and, hence, in $\hat{G}$. $\hat{G}$, being a subgroup of H, is an O-group, and so $\left[a, f_1 f_2^{-1}\right] \neq 1$ in $\hat{G}$. This means that $f_1^{-1} a f_1 \neq f_2^{-1} a f_2$ for some a in A (and hence in $\hat{A}$). So, f_1 and f_2 induce distinct automorphisms of $\hat{A}$. Finally, $\Phi^{\#}$ is an O-automorphism group of $\hat{A}$ because $\hat{G}$ is an O-group (0.3 Theorem 6).

It remains to show that $\hat{A}$ is a normal, divisible, abelian subgroup of $\hat{G}$ and that $\hat{A} \cap \Phi^{\#} = \{1\}$. To show that $\hat{A}$ is abelian, take any a_1, a_2 in A and f in $\Phi^{\#}$ with f^m in Φ (m in Z^+). Then

$$1 = \left[f^{-m} a_1 f^m, a_2\right] = \left[a_1, f^m, a_2\right]$$

(since A is an abelian subgroup of G). By 1.3 Theorem 3, $[a_1, f, a_2] = 1$; that is, $\left[f^{-1} a_1 f, a_2\right] = 1$. So $\hat{A}$ is abelian, since for all a_1, a_2 in A and f_1, f_2 in $\Phi^{\#}$,

$$\left[f_1^{-1} a_1 f_1, f_2^{-1} a_2 f_2\right] = f_2^{-1}\left[\left(f_1 f_2^{-1}\right)^{-1} a_1 f_1 f_2^{-1}, a_2\right] f_2 = 1 .$$

$\hat{A}$ is normal in $\hat{G}$ by the definitions of $\hat{A}$ and $\hat{G}$, and $\hat{A}$ is divisible since A is divisible and $\hat{A}$ is abelian. To show that $\hat{A} \cap \Phi^{\#} = \{1\}$, suppose g (in $\hat{G}$) is in $\hat{A}$ and $\Phi^{\#}$. Then there is m in Z^+ such that g^m is in Φ. So g^m is in $\hat{A} \cap \Phi$. Since g^m is in $\hat{A}$, it follows, in particular, that $[g^m, a] = 1$ for all a in A. So g^m, regarded as an element of Φ, an automorphism group of A, is the identity automorphism; that is $g^m = 1$ in Φ. Now $\Phi^{\#}$ is torsion-free, so $g = 1$ in $\Phi^{\#}$ and, hence, in $\hat{G}$. //

(We remind the reader that for the remainder of this chapter, the notation introduced in Section 3.3 will be used. That is, A is a divisible, abelian O-group written *additively* with abelian O-automorphism group, Φ, written *multiplicatively*.)

4.2 The solution (when it exists!) for a weaker proposition

For n in Z^+, consider the following proposition:-

$\underline{P_n}$ *Let* A *be a divisible, abelian* O*-group and let* Φ *be an abelian* O*-automorphism group of* A *such that there is* ψ *in* Φ *with no* n*-th root in* Φ. *Then* A *can be extended to a divisible, abelian* O*-group,* $\hat{A}$, *satisfying:-*

(i) each ϕ *in* Φ *can be extended to* $\hat{\phi}$ *in* $\mathrm{aut}(\hat{A})$,

(ii) the mapping $\phi \mapsto \hat{\phi}$ *is an isomorphism from* Φ *into* $\mathrm{aut}(\hat{A})$,

(iii) there is θ *in* $\mathrm{aut}(\hat{A})$ *such that* $\theta^n = \hat{\psi}$, *and* $\mathrm{gp}(\hat{\Phi}, \theta)$ *is an abelian* O*-automorphism group of* $\hat{A}$ *(where* $\hat{\Phi} = \{\hat{\phi} : \phi \in \Phi\}$*).*

The usual transfinite arguments show that if P_n is true for all n in Z^+, then P is true.

In this section, and the two subsequent sections, we shall discuss P_2 . Hopefully, if P_2 is true, then P_n $(n > 2)$ will follow with at least reasonable ease.

Suppose A, Φ and ψ satisfy the hypotheses of P_2 . If ψ has a square root, θ , in $\mathrm{aut}(A)$ such that θ centralizes Φ and $\mathrm{gp}(\Phi, \theta)$ is an O-automorphism group of A , then we simply let $\hat{A} = A$ and $\hat{\phi} = \phi$ for all ϕ in Φ .

So, we shall suppose that for any square root, χ , of ψ in $\mathrm{aut}(A)$ (and there may be many, as in the example in 4.4 to follow), $\mathrm{gp}(\Phi, \chi)$ is either not abelian *or* not an O-automorphism group of A .

Assume, for the moment, that a suitable extension, $\hat{A}$, of A exists. (Here, and in the sequel, we mean by a *suitable extension*, one which satisfies the conclusions of P_2 .) We want to determine what $\hat{A}$ looks like.

Let $A_1 = \{a+a'\theta : a, a' \in A\}$ (where θ is given by P_2).

LEMMA 1. *A_1 is a suitable extension of A .*

Proof. Clearly, A_1 is a subgroup of $\hat{A}$. For all a, a' in A , ϕ in Φ , and m in Z^+ ,

$$\begin{aligned}(1/m)(a+a'\theta) &= (1/m)a + (1/m)(a'\theta)\\ &= (1/m)a + \big((1/m)a'\big)\theta \quad \text{which is in } A_1\end{aligned}$$

(since A is divisible); so, A_1 is divisible.

$$\begin{aligned}(a+a'\theta)\hat{\phi} &= a\hat{\phi} + a'\theta\hat{\phi}\\ &= a\phi + a'\phi\theta \quad \big(\text{by } (i) \text{ and } (iii) \text{ of } P_2\big), \text{ which is in } A_1 .\end{aligned}$$

So, A_1 is $\hat{\phi}$-invariant for all ϕ in Φ . Also,

$$\begin{aligned}(a+a'\theta)\theta &= a\theta + a'\theta^2\\ &= a'\psi + a\theta \quad \big(\text{by } (iii) \text{ and } (i) \text{ of } P_2\big), \text{ which is in } A_1 ;\end{aligned}$$

so, A_1 is θ-invariant.

It follows easily from the ϕ-invariance (for all ϕ in Φ) and θ-invariance of A_1 that A_1 is a suitable extension. //

Henceforth, we shall suppose that $\hat{A}$ is simply A_1 . So, $\hat{A}$ is the (not necessarily direct) sum, $A + A\theta$. Clearly, the sum is direct if, and only if, $a \neq 0$ in A implies $a\theta \notin A$. This situation need not arise. For example (remembering the vector space notation introduced in Section 3.3 p. 57), if A is $VS(1, \sqrt{2}, \sqrt{3})$, a subspace of R , and $\Phi = gp(\psi)$ where ψ is the automorphism given by $a\psi = 2a$ for all a in A , then we may take $\hat{A}$ to be $VS(1, \sqrt{2}, \sqrt{3}, \sqrt{6})$ and $\theta : \hat{A} \to \hat{A}$ given by

$$(\alpha+\beta\sqrt{2}+\gamma\sqrt{3}+\delta\sqrt{6})\theta = 2\beta + \alpha\sqrt{2} + 2\delta\sqrt{3} + \gamma\sqrt{6} .$$

So, $1 + \sqrt{2} \neq 0$ and $(1+\sqrt{2})\theta = 2 + \sqrt{2}$ is in A .

To find a convenient, direct sum representation of $\hat{A}$, we define

$$B = \{a \in A : a\theta \in A\} . \qquad (4.2.1)$$

LEMMA 2. *B is a divisible subgroup of A and is $\hat{\phi}$-invariant for all ϕ in Φ and θ-invariant.*

Proof. The fact that B is a divisible subgroup of A is shown easily. To show that B is $\hat{\phi}$-invariant for all ϕ in Φ , take any b in B and ϕ in Φ . Then $b\hat{\phi}\theta = b\theta\hat{\phi} = b\theta\phi$ (since $b\theta$ is in A); so, $b\hat{\phi}\theta$ is in A whence $b\hat{\phi}$ is in B . For θ-invariance, we have $(b\theta)\theta = b\theta^2 = b\hat{\psi} = b\psi$. That is, $(b\theta)\theta$ is in A , whence $b\theta$ is in B . //

Since B is divisible, it is a direct summand of A , and, so we can write $A = B + C$, a direct sum, where $B \cap C = \{0\}$ (see Fuchs [6], p. 62).

We know that any element of $\hat{A}$ is of the form $a + a'\theta$. Setting $a = b + c$ and $a' = b' + c'$, where b, b' are in B and c, c' are in C , we have

$$\begin{aligned} a + a'\theta &= b + c + (b'+c')\theta \\ &= b + b'\theta + c + c'\theta . \end{aligned}$$

Since B is θ-invariant, it follows that $\hat{A} = B + C + C\theta$ and, furthermore, we have:-

THEOREM 2. *$\hat{A} = B + C + C\theta$ where the sum is direct.*

Proof. Take $b + c + c'\theta = 0$ with b in B, and c, c' in C. Then $c'\theta = -(b+c)$ is in A; whence, c' is in B [by (4.2.1)]. That is, c' is in $B \cap C = \{0\}$. So, $b + c = 0$ and, since $B + C$ is direct, we have $b = c = 0$. //

4.3 A step towards a solution

We know what $\hat{A}$ will be (when it exists), but still have the problem of constructing $\hat{A}$ given A, Φ and ψ. Clearly, we have to determine which elements of A *must* have their image under the (proposed) automorphism, θ, already in A. We shall see (in Section 4.4) that a very obvious choice of square root for ψ need not work. Furthermore, the same example will show that distinct minimal suitable extensions, $\hat{A}_1 = B_1 + C_1 + C_1\theta_1$ and $\hat{A}_2 = B_2 + C_2 + C_2\theta_2$, can exist with $B_1 \neq B_2$ in A.

Here, we present a subgroup, B_0 of A, which must be θ-invariant for all suitable extensions (when they exist). Let

$$B_0 = \mathrm{gp}\{a \in A : a\chi^2 = a\psi \text{ for some } \chi \text{ in } \Phi\} .$$

We shall denote elements of B_0 by $a_1 + a_2 + \ldots + a_k$ where $a_j\chi_j^2 = a_j\psi$ for $j = 1, 2, \ldots, k$. (Whenever in the sequel we "take $a_1 + a_2 + \ldots + a_k$ in B_0" we shall leave unsaid, but nevertheless assume, that there are corresponding χ_j in Φ such that $a_j\chi_j^2 = a_j\psi$ for $j = 1, 2, \ldots, k$.)

LEMMA 3. *B_0 is ϕ-invariant for all ϕ in Φ.*

Proof. Take any $b = a_1 + a_2 + \ldots + a_k$ in B_0. Then $b\phi = a_1\phi + a_2\phi + \ldots + a_k\phi$. Since Φ is abelian, we have for each $j = 1, 2, \ldots, k$,

$$(a_j\phi)\chi_j^2 = \left(a\chi_j^2\right)\phi = (a\psi)\phi = (a\phi)\psi \, . \qquad (4.3.1)$$

So, each $a_j\phi$ is in (the set which generates) B_0 ; whence, $b\phi$ is in B_0 . //

B_0 has an automorphism which is a square root of ψ_0 (the restriction to B_0 of ψ). Define $\theta_0 : B_0 \to B_0$ by

$$\left(a_1 + a_2 + \dots + a_k\right)\theta_0 = a_1\chi_1 + a_2\chi_2 + \dots + a_k\chi_k \, . \qquad (4.3.2)$$

The following lemma shows that this mapping is single-valued.

LEMMA 4. *Suppose* a *in* A *and* ϕ, χ *in* Φ *satisfy* $a\phi^2 = a\chi^2$. *Then* $a\phi = a\chi$.

Proof. Choose an order, $\le$, of A with respect to which Φ is an o-automorphism group of A . (Such an order exists because A and Φ satisfy the hypotheses of P_2 .) If $a\phi < a\chi$, we have (since Φ is abelian), $a\phi^2 < (a\chi)\phi = (a\phi)\chi < a\chi^2$. Similarly, if $a\phi > a\chi$, then $a\phi^2 > a\chi^2$. //

We can say more about θ_0 ; namely:-

LEMMA 5. *If* $\le$ *is an order of* A *with respect to which* Φ *is an* o-*automorphism group of* A , *then* θ_0 *is an* o-*automorphism of* B_0 (*the order of* B_0 *being that induced by the order of* A) *and* $\theta_0^2 = \psi_0$.

Proof. It is a straightforward matter to show that θ_0 is a homomorphism. To show that θ_0 is onto, take any $a_1 + a_2 + \dots + a_k$ in B_0 . By (4.3.1), each $a_j\chi_j^{-1}$ is in the generating set of B_0 and $\left(a_1\chi_1^{-1} + a_2\chi_2^{-1} + \dots + a_k\chi_k^{-1}\right)\theta_0 = a_1 + a_2 + \dots + a_k$ (by the definition, (4.3.2), of θ_0).

Next, we show that θ_0 preserves order. Take $b = a_1 + a_2 + \dots + a_k > 0$ in B_0 . The proof is by induction on k . If $k = 1$, then $b = a_1 > 0$

and $b\theta_0 = a_1\chi_1 > 0$ (since χ_1 is in Φ, an o-automorphism group of A). The induction hypothesis is that for all $b' = a_1 + a_2 + \ldots + a_{k-1}$ $(k > 1)$ in B_0, we have $b' > 0$ implies $b'\theta_0 > 0$. By way of contradiction, we assume that $b\theta_0 = a_1\chi_1 + a_2\chi_2 + \ldots + a_k\chi_k \leq 0$, and our contradiction is achieved by showing that $b\theta_0 > 0$. It follows that $b\theta_0 \nleq 0$ and, since $(A, \leq)$ is an o-group, we have $b > 0$ implies $b\theta_0 > 0$.

Now $b\theta_0 \leq 0$ implies that $a_2\chi_2 + \ldots + a_k\chi_k \leq -a_1\chi_1$. Remembering (throughout this proof) that Φ is abelian and that $a_j\chi_j^2 = a_j\psi$ for all $j = 1, 2, \ldots, k$, we have

$$a_2\chi_1\chi_2 + \ldots + a_k\chi_1\chi_k \leq -a_1\chi_1^2 = -a_1\psi < (a_2 + \ldots + a_k)\psi$$

(since $b > 0$ implies $a_2 + \ldots + a_k > -a_1$ and ψ is an o-automorphism). Since $(a_2 + \ldots + a_k)\psi = a_2\chi_2^2 + \ldots + a_k\chi_k^2$, we have

$$(a_2\chi_1 - a_2\chi_2)\chi_2 + \ldots + (a_k\chi_1 - a_k\chi_k)\chi_k < 0 . \qquad (4.3.3)$$

Now, for all $j = 2, \ldots, k$, we have

$$\begin{aligned}(a_j\chi_1 - a_j\chi_j)\chi_j^2 &= a_j\chi_j^2\chi_1 - a_j\chi_j^2\chi_j \\ &= (a_j\chi_1 - a_j\chi_j)\psi .\end{aligned}$$

So, we can apply the (logical negation of the) induction hypothesis to (4.3.3). This gives

$$(a_2\chi_1 - a_2\chi_2) + \ldots + (a_k\chi_1 - a_k\chi_k) \leq 0 .$$

It follows that $-a_1\chi_1 < (a_2 + \ldots + a_k)\chi_1 \leq a_2\chi_2 + \ldots + a_k\chi_k$ and, so, $b\theta_0 > 0$.

So we have shown that θ_0 is an o-automorphism of B_0.

To complete the proof of the lemma, we must show that $\theta_0^2 = \psi_0$ (where

ψ_0 denotes the restriction to B_0 of ψ). By Lemma 3, of this section, B_0 is ψ-invariant and, so, ψ_0 exists. Take any $b = a_1 + a_2 + \ldots + a_k$ in B_0. Then

$$
\begin{aligned}
b\theta_0^2 &= (a_1\chi_1 + a_2\chi_2 + \ldots + a_k\chi_k)\theta_0 \\
&= a_1\chi_1^2 + a_2\chi_2^2 + \ldots + a_k\chi_k^2 \quad (cf.\ (4.3.1)) \\
&= a_1\psi + a_2\psi + \ldots + a_k\psi \\
&= b\psi \\
&= b\psi_0 \ . \quad //
\end{aligned}
$$

As a corollary to this we have:-

COROLLARY 1. *Suppose that* A *has an extension,* $\hat{A}$ *, satisfying (i), (ii) and (iii) of* P_2 *. Then* $B_0 \leq B$ *, where* B *is given by* (4.2.1).

Proof. If $a\phi^2 = a\psi$ for a in A and ϕ in Φ , we have $a\hat{\phi}^2 = a\phi^2 = a\psi = a\theta^2$; whence, by Lemma 4 of this section, $a\hat{\phi} = a\theta$, and so $a\phi = a\theta$. So, for $b = a_1 + a_2 + \ldots + a_k$ in B_0 ,

$$
\begin{aligned}
b\theta &= a_1\theta + a_2\theta + \ldots + a_k\theta \\
&= a_1\chi_1 + a_2\chi_2 + \ldots + a_k\chi_k \\
&= b\theta_0 \quad (\text{by } (4.3.2)) .
\end{aligned}
$$

Hence, for all b in B_0 , $b\theta = b\theta_0$ is in A and, so, b is in B by (4.2.1) . That is, $B_0 \leq B$. //

The question asked at this stage is:-

Given A, Φ and ψ satisfying the hypotheses of P_2 , can a suitable extension, $\hat{A}$, be constructed in which $B = B_0$?

The answer to this question is, I suspect, "not always".*

* See Section 4.5.

4.4 An example

4.4.1. The example presented illustrates some of the difficulties encountered in trying to prove (or disprove) P_2. It also has the interesting property that two non-isomorphic, suitable extensions exist. This means that if Kokorin's question can be answered "yes", then metabelian, orderable d-closures of an MO-group need not be unique up to isomorphism (unlike the nilpotent case - see Mal'cev [18]).

Throughout this section, all vector spaces of the form $\mathrm{VS}(v_1, v_2)$ (over Q) will be isomorphic to the subspace, $\mathrm{VS}(1, \sqrt{2})$, of R. The isomorphism will be that induced by $v_1 \mapsto 1$ and $v_2 \mapsto \sqrt{2}$. $\mathrm{VS}(v_1, v_2)$ will always have the natural order of $\mathrm{VS}(1, \sqrt{2})$. That is, $r_1v_1 + r_2v_2 > 0$ if, and only if, $r_1 + r_2\sqrt{2} > 0$ in R, for r_1, r_2 in Q.

Now for the example itself. Let $A = X + Z$ where $X = \mathrm{VS}(x_1, x_2)$ and $Z = \mathrm{VS}(z_1, z_2)$. Φ is the group generated by ϕ and ψ where, for all $a = x + z$ in A (x in X and z in Z), ϕ and ψ are given by

$$a\psi = 2a ,$$

and

$$a\phi = (x+z\sigma) + z ,$$

where $\sigma = Z \to X$ is the homomorphism given by

$$(r_1z_1+r_2z_2)\sigma = (r_1+r_2)x_1 \quad \text{for all } r_1, r_2 \text{ in } \mathsf{Q} . \tag{4.4.1}$$

Writing ψ and ϕ as matrices (see, for example, Fuchs [6], pp. 212-213, or, in particular, Conrad [4], pp. 382-383, where he discusses the representation of o-automorphisms by matrices), we have

$$\psi = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \phi = \begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix} .$$

These matrices are non-singular, whence ψ and ϕ are automorphisms.

(ψ is clearly an automorphism anyway since A is divisible, abelian and torsion-free.) Φ is abelian since all automorphisms of A of the form $a \longmapsto ra$ (for some r in $\mathbb{Q}$) are central in $\mathrm{aut}(A)$. A is an O-group, so it remains to show that Φ is an O-automorphism group of A . ψ is an o-automorphism for *all* orders of A , and the following order of A makes ϕ an o-automorphism:-

$$a = x + z > 0 \text{ if, and only if, } z > 0 ,$$
$$\textit{or } z = 0 \text{ and } x > 0 .$$

It is easy to see that ϕ preserves this order.

We shall consider the adjunction of a square root to ψ . (Remember that, amongst other things, we want the square root to centralize Φ .) To show that suitably adjoining a square to ψ is no trivial matter, we have:-

LEMMA 6. *If* $\chi^2 = \psi$ *for some* χ *in* $\mathrm{aut}(A)$, *then* $[\chi, \phi] \neq 1$.

Proof. We write χ as the (non-singular) matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, where α and δ are endomorphisms of X and Z respectively, and β (respectively γ) is a homomorphism from X into Z (respectively Z into X). Since $\chi^2 = \psi$, we have, in particular, that

$$\alpha^2 + \beta\gamma = 2 \tag{4.4.2}$$

(where 2 represents here the automorphism $x \longmapsto 2x$ of X). It is straightforward to show that $(x+z)(\chi\phi) = (x+z)(\phi\chi)$ for all $x + z$ in A if and only if,

$$(x\beta+z\delta)\sigma = (z\sigma)\alpha \tag{4.4.3}$$

and

$$(z\sigma)\beta = 0 \tag{4.4.4}$$

for all x in X and z in Z .

Assume, by way of contradiction, that $a(\chi\phi) = a(\phi\chi)$ for all a in A .

The endormophisms, α and δ , may be written as 2×2 matrices with rational coefficients (since X and Z are simply the direct sum of two copies of the rationals). So, we have

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

and

$$\delta = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} .$$

Furthermore, β can be regarded as a 2×2 matrix with rational coefficients. We have

$$\begin{aligned}(r_1x_1+r_2x_2)\beta &= (r_1x_1+r_2x_2)\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \\ &= (r_1\beta_{11}+r_2\beta_{21})z_1 + (r_1\beta_{12}+r_2\beta_{22})z_2 .\end{aligned}$$

Now (4.4.4) implies that, for all r in Q, $(rx_1)\beta = 0$ (see (4.4.1)). From this, it follows that $(r\beta_{11})z_1 + (r\beta_{12})z_2 = 0$ for all r in Q, whence

$$\beta_{11} = \beta_{12} = 0 . \tag{4.4.5}$$

Setting $x = 0$ in (4.4.3), we have $(z\delta)\sigma = (z\sigma)\alpha$ for all z in Z. This implies that, for all r_1, r_2 in Q,

$$\big(r_1(\delta_{11}+\delta_{12})+r_2(\delta_{21}+\delta_{22})\big)x_1 = (r_1+r_2)\alpha_{11}x_1 + (r_1+r_2)\alpha_{12}x_2 ,$$

whence $\alpha_{12} = 0$.

So

$$\alpha = \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & \alpha_{22} \end{pmatrix} .$$

By (4.4.5),

$$\beta = \begin{pmatrix} 0 & 0 \\ \beta_{21} & \beta_{22} \end{pmatrix} .$$

Substituting these in (4.4.2) implies, in particular, that $\alpha_{11}^2 = 2$. α_{11} is rational, so this is impossible and we have the required contradiction. //

4.4.2. The first of our two non-isomorphic, suitable extensions of A has $B = X$ (where B is given by (4.2.1)).

Let $\hat{A} = X + Y + Z$ where X and Z are as before and $Y = \mathrm{VS}(y_1, y_2)$.

Observe that if $V = \mathrm{VS}(v_1, v_2)$, then the automorphism, $\sqrt{2}$, of V which sends $r_1v_1 + r_2v_2$ to $2r_2v_1 + r_1v_2$ is a square root of the automorphism which sends v in V to $2v$. We define θ in $\mathrm{aut}(\hat{A})$ in such a way that $x_1\theta = x_2 = x_1\sqrt{2}$, $x_2\theta = 2x_1 = x_2\sqrt{2}$, $y_1 = \frac{1}{2}(2z_1-z_2\theta) = \frac{1}{2}(z_2\sqrt{2}-z_2\theta)$ and $y_2 = z_1\theta - z_2 = z_1\theta - z_1\sqrt{2}$. In fact,

$$\theta = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & \theta' & \sqrt{2} \end{pmatrix} ,$$

where $\theta' : Z \to Y$ is the homomorphism

$$(r_1z_1+r_2z_2)\theta' = -2r_2y_1 + r_1y_2 . \tag{4.4.6}$$

It is a straightforward matter to show that $\theta^2 = \hat{\psi}$ where $\hat{\psi}$ is represented by the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

Finally we define

$$\hat{\phi} = \begin{pmatrix} 1 & 0 & 0 \\ \phi' & 1 & 0 \\ \sigma & 0 & 1 \end{pmatrix}$$

where σ is given by (4.4.1) and $\phi' : Y \to X$ is the homomorphism

$$(r_1y_1+r_2y_2)\phi' = (r_1-r_2)x_1 + (r_2-r_1/2)x_2 . \tag{4.4.7}$$

Again, it is straightforward to show that $[\hat{\phi}, \theta] = 1$, whence $\hat{\Phi} = gp(\hat{\phi}, \theta)$ is abelian. To show that $\hat{\Phi}$ is an O-automorphism group of $\hat{A}$, we merely observe that under the following order of $\hat{A}$, both $\hat{\phi}$ and θ are o-automorphisms:-

$$\begin{aligned} x + y + z > 0 \text{ if, and only if, } & z > 0 , \\ \text{or } & z = 0 \text{ and } y > 0 , \\ \text{or } & z = 0 , \ y = 0 \text{ and } x > 0 . \end{aligned}$$

So, *(iii)* of P_2 is satisfied. The verification of *(i)* and *(ii)* of P_2 is tedious, although straightforward, and is omitted.

4.4.3. The second suitable extension of A has $B = \{0\}$.

Let $\hat{A} = W + X + Y + Z$ where X, Y and Z are as before and $W = \mathrm{VS}(w_1, w_2)$. This time, we define θ so that $w_1 = \frac{1}{4}(x_2\theta - x_2\sqrt{2})$, $w_2 = \frac{1}{2}(x_1\sqrt{2} - x_1\theta)$, $y_1 = \frac{1}{2}(z_2\sqrt{2} - z_2\theta)$ and $y_2 = z_1\theta - z_1\sqrt{2}$.

$$\theta = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ \theta'' & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & \theta' & \sqrt{2} \end{pmatrix}$$

where θ' is given by (4.4.6) and $\theta'' : X \to W$ is the homomorphism

$$(r_1x_1 + r_2x_2)\theta'' = 4r_2w_1 - 2r_1w_2 .$$

Again,

$$\hat{\psi} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} .$$

Define

$$\hat{\phi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \phi'' & \phi' & 1 & 0 \\ 0 & \sigma & 0 & 1 \end{pmatrix}$$

where σ is given by (4.4.1), ϕ' is given by (4.4.7) and $\phi'' : Y \to W$ is the homomorphism

$$(r_1y_1+r_2y_2)\phi'' = (r_1-2r_2)w_2 \ .$$

Both θ and $\hat{\phi}$ are o-automorphisms under the order of $\hat{A}$ given by $w+x+y+z > 0$ if, and only if, $z > 0$,

$$\begin{aligned} &or \quad z = 0 \ \text{ and } \ y > 0 \ , \\ &or \quad z = 0 \ , \ y = 0 \ \text{ and } \ x > 0 \ , \\ &or \quad z = 0 \ , \ y = 0 \ , \ x = 0 \ \text{ and } \ w > 0 \ . \end{aligned}$$

Once again, the verification that $\hat{A}$ is a suitable extension of A is straightforward, but tedious, and is omitted.

4.5 Addendum

The time spent writing a thesis can be one of extreme frustration, especially if a problem remains as tantalizingly open as the problem considered in this chapter. It has taken some effort of will to persist with the writing rather than tackle the problem. Of course, there have been weak moments, and in one such moment, the thought occurred that if any metabelian *NGP*-group is an *O*-group, then P_2 (and hopefully P_n for $n > 2$) may be proven more easily. Imagine my delight, when checking details of the bibliography, to read B. Gordon's review of a paper of Kokorin [15]. Kokorin remarks, in this paper, that any metabelian *NGP*-group is an *O*-group.

There followed, almost immediately, a solution for P_2 in the case where $B_1 = \{0\}$. (B_1 is a generalization of the subgroup, B_0 , defined in Section 4.3.) Below we outline proofs of both Kokorin's result and the partial solution of P_2 .

THEOREM 3 (Kokorin [15]). *If G is a metabelian NGP-group, then G is an O-group.*

Proof. Assume, by way of contradiction, that G is not orderable. Let $A = I(G')$ and let P, a subset of A, be a maximal, partial order of G. Since G is an NGP-group, $\{1\} \neq P$. Also, A is abelian (by 2.1 Lemma 2, the remark preceding Section 2.1 and the easily verified fact that any NGP-group is an R-group). Furthermore, since G is not orderable, P is not a full order of A. It can be shown that P is strongly isolated in G. That is, for $g, g_1, g_2, \ldots, g_k$ in G, if $g^{g_1} g^{g_2} \ldots g^{g_k}$ is in P, then g is in P. From this, it follows that P must be a full order of A, and this is the required contradiction. //

Now suppose that A, Φ and ψ satisfy the hypotheses of P_2. In view of 3.3 Theorem 3, we suppose that for all $\phi_1, \phi_2, \ldots, \phi_n$ in Φ, the endomorphism $\phi_1 + \phi_2 + \ldots + \phi_n$ is onto. B_1 is the subgroup of A generated by those a in A which have the property $a\nabla^2 = a\psi$, where ∇ is a product of automorphisms of the form $(\phi_1 + \phi_2 + \ldots + \phi_n)^{\pm 1}$ for some $\phi_1, \phi_2, \ldots, \phi_n$ in Φ.

THEOREM 4. *If $B_1 = \{0\}$, then a suitable extension, $\hat{A}$, of A exists.*

Proof. Let $\{a_i : i \in I\}$ be a basis of A as a vector space over $\mathbb{Q}$. Let D be the vector space with basis $\{d_i : i \in I\}$, where each d_i is a new symbol. $\iota : A \to D$ is the isomorphism from A onto D induced by the mapping, $a_i \mapsto d_i$, of $\{a_i : i \in I\}$ onto $\{d_i : i \in I\}$. $\hat{A}$ is the direct sum $A + D$ and for all ϕ in Φ, we define $\hat{\phi} : \hat{A} \to \hat{A}$ by

$$(a+d)\hat{\phi} = a\phi + \iota\left(\left(\iota^{-1}(d)\right)\phi\right) . \tag{4.5.1}$$

It is reasonably straightforward to show that *(i)* and *(ii)* of P_2 are satisfied. For *(iii)*, we define $\theta : \hat{A} \to \hat{A}$ by

$$(a+d)\theta = \left(\iota^{-1}(d)\right)\psi + \iota(a) . \tag{4.5.2}$$

(This definition makes $D = A\theta$.) Once again, it is not difficult to show that θ is an automorphism of $\hat{A}$, $\theta^2 = \hat{\psi}$ and $[\theta, \hat{\phi}] = 1$ for all ϕ in Φ . Writing ϕ for $\hat{\phi}$ and Φ for $\hat{\Phi}$, we show that $\Phi_1 = \mathrm{gp}(\Phi, \theta)$ is an O-automorphism group of $\hat{A}$ by showing that $\hat{F} = \hat{A}\lambda\Phi_1$ is an NGP-group. By Theorem 3 of this section and 0.3 Theorem 6, the required result follows.

Elements of Φ_1 can be written uniquely in the form $\phi\theta^{\varepsilon}$ where ϕ is in Φ and $\varepsilon = 0$ or 1 . Suppose that

$$(\phi\theta^{\varepsilon}, \hat{a})^{\left(\phi_1\theta^{\varepsilon_1}, \hat{a}_1\right)}(\phi\theta^{\varepsilon}, \hat{a})^{\left(\phi_2\theta^{\varepsilon_2}, \hat{a}_2\right)} \dots (\phi\theta^{\varepsilon}, \hat{a})^{\left(\phi_n\theta^{\varepsilon_n}, \hat{a}_n\right)} = 1$$

for some $(\phi\theta^{\varepsilon}, \hat{a})$ and $\left(\phi_j\theta^{\varepsilon_j}, \hat{a}_j\right)$ $(j = 1, 2, \dots, n)$ in $\hat{F}$. We deduce, immediately, that $(\phi\theta^{\varepsilon})^n$, and hence $\phi\theta^{\varepsilon}$, equals 1 . It follows that $\hat{a}\phi_1\theta^{\varepsilon_1} + \hat{a}\phi_2\theta^{\varepsilon_2} + \dots + \hat{a}\phi_n\theta^{\varepsilon_n} = 0$. Since $\hat{A}$ is abelian, we may suppose that $\varepsilon_j = 0$ for $n = 1, 2, \dots, m$ and $\varepsilon_j = 1$ for $j = m+1, m+2, \dots, n$, where $1 \le m \le n$. From the definitions (4.5.1) and (4.5.2), and the fact that $A + D$ is direct, we have

$$a\Delta_1 + \left(\left(\iota^{-1}(d)\right)\psi\right)\Delta_2 = 0 \quad \text{and} \quad \left(\iota^{-1}(d)\right)\Delta_1 + a\Delta_2 = 0 \quad \text{in } A ,$$

where $\Delta_1 = \phi_1 + \phi_2 + \dots + \phi_m$ and $\Delta_2 = \phi_{m+1} + \phi_{m+2} + \dots + \phi_n$. These equations give

$$\left(\left(\iota^{-1}(d)\right)\psi\right)\Delta_2^2 = -a\Delta_1\Delta_2 = -a\Delta_2\Delta_1 = \left(\iota^{-1}(d)\right)\Delta_1^2$$

That is,

$$\left(\iota^{-1}(d)\right)\psi = \left(\iota^{-1}(d)\right)\Delta_1^2\Delta_2^{-2} = \left(\iota^{-1}(d)\right)\left(\Delta_1\Delta_2^{-1}\right)^2 .$$

This means that $\iota^{-1}(d)$ is in $B_1 = \{0\}$; so $d = 0$. Whence, $a = 0$ and so $\hat{a} = 0$. So $(\phi\theta^{\varepsilon}, \hat{a}) = 1$ and $\hat{F}$ is an NGP-group. //

APPENDIX 1

A SUPPLEMENTARY EXAMPLE

The group E_0 (described below) has, as a homomorphic image, the group, E, of Chapter 1. The generating set of E_0 is $\{x_{r,i}, y_{r,i}, z : r, i \in \mathbb{Z}\}$ and is best laid out as follows:-

$$
\begin{array}{ccccc}
 & & z & & \\
 & \vdots & \vdots & \vdots & \\
\ldots, & y_{1,-1}, & y_{1,0}, & y_{1,1}, & \ldots \\
\ldots, & x_{1,-1}, & x_{1,0}, & x_{1,1}, & \ldots \\
 & & & & \\
\ldots, & y_{0,-1}, & y_{0,0}, & y_{0,1}, & \ldots \\
\ldots, & x_{0,-1}, & x_{0,0}, & x_{0,1}, & \ldots \\
 & & & & \\
\ldots, & y_{-1,-1}, & y_{-1,0}, & y_{-1,1}, & \ldots \\
\ldots, & x_{-1,-1}, & x_{-1,0}, & x_{-1,1}, & \ldots \\
 & \vdots & \vdots & \vdots &
\end{array}
$$

Let T be an arbitrary, fixed subset of $\mathbb{Z}^+$. The defining relations for E_0 are:-

For all r, i, j in $\mathbb{Z}$ and s, k in $\mathbb{Z}^+$,

(1) $$y_{r,i}^{-1}x_{r,i}y_{r,i} = x_{r,i}^2 ;$$

(2) $$y_{r,i+k}^{-1} y_{r,i} y_{r,i+k} = \begin{cases} y_{r,i} x_{r,i}^{-1} & \text{if } k \in T , \\ y_{r,i} & \text{if } k \notin T ; \end{cases}$$

$$y_{r,i+k}^{-1} x_{r,i} y_{r,i+k} = x_{r,i} ,$$

$$x_{r,i+k}^{-1} y_{r,i} x_{r,i+k} = x_{r,i} ,$$

$$x_{r,i+k}^{-1} x_{r,i} x_{r,i+k} = x_{r,i} ;$$

(3) $$y_{r+s,j}^{-1} y_{r,i} y_{r+s,j} = \begin{cases} y_{r,i+1} & \text{if } s = 1 , \\ y_{r,i} & \text{if } s \geq 2 ; \end{cases}$$

$$y_{r+s,j}^{-1} x_{r,i} y_{r+s,j} = \begin{cases} x_{r,i+1} & \text{if } s = 1 , \\ x_{r,i} & \text{if } s \geq 2 , \end{cases}$$

$$x_{r+s,j} y_{r,i} x_{r+s,j} = y_{r,i} ,$$

$$x_{r+s,j} x_{r,i} x_{r+s,j} = x_{r,i} ;$$

(4) $$z^{-1} y_{r,i} z = y_{r+1,i} ,$$

$$z^{-1} x_{r,i} x = x_{r+1,i} .$$

Observe that in E (see Chapter 1, p. 24) $a_r(i, 0, 1)$, $a_r(i, 1, 0)$ and e satisfy the relations (1), (2), (3) and (4) by (respectively) (1.2.2), (1.2.5), (1.2.10) and the definition, $d^e = d\phi'$, preceding 1.2 Lemma 4, where $a_r(i, 0, 1)$, $a_r(i, 1, 0)$ and e replace $x_{r,i}$, $y_{r,i}$ and z respectively in (1), (2), (3) and (4). So E is a homomorphic image of E_0 .

APPENDIX 2

AN ACKNOWLEDGEMENT

Holland [11] proved the result contained in the appended paper [5] and announced the existence of "an R-group which is not an O-group" in [10] (1960). Hollister [12] proved a (possibly) stronger result for the same group, G (with presentation $\{a, c; a^{-1}c^2a = c^2a^2c^2\}$). He shows that the element $acaca^{-1}c^{-1}$ is generalized periodic, and so G is not an NGP-group. (This result is only *possibly* stronger because the question of whether or not any NGP-group is an O-group appears to be open. See [14], question 1.47.)

I thank A.M.W. Glass for drawing my attention to these results of Holland and Hollister.

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On a problem in the theory of ordered groups

Colin D. Fox

The group G presented on two generators a, c with the single defining relation $a^{-1}c^2a = c^2a^2c^2$ [proposed by B.H. Neumann in 1949 (unpublished), discussed by Gilbert Baumslag in *Proc. Cambridge Philos. Soc.* 55 (1959)] has been considered as a possible example of an orderable group which can not be embedded in a divisible orderable group, contrary to the conjecture that no such examples exist. It is known from Baumslag's discussion that G can not be embedded in any divisible orderable group. However, it is shown in this note that G is not orderable, and thus is not a counter-example to the conjecture.

DEFINITIONS. A group, G, is an orderable group (0-group) if G admits a linear order, $\leq$, which has the property that if $x \leq y$ then $axb \leq ayb$ for a, b, x, y in G.

G is an R-group if it has the property that $x^n = y^n$ implies $x = y$ for x, y in G.

G is a divisible group if for each g in G and integer, n, there exists a (not necessarily unique) x in G such that $x^n = g$.

It is convenient to ignore the presentation of the group G given in the abstract, and instead to construct G as a generalized free product, as follows:

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Take groups $A = \mathrm{gp}(a, b : a^{-1}ba = b^2)$ (whose elements may be written uniquely in the form $a^n b^\beta$ where n is an integer and $\beta = m2^{-k}$, m an integer and k a non-negative integer - see Fuchs [2], p. 60) and C, the infinite cyclic group with generator, c. Let G be the generalized free product of A and C with amalgamated subgroup

$$H = \mathrm{gp}(c^2 = ba^{-2}) = \mathrm{gp}(c^2 = a^{-2}b^{1/4}) .$$

Baumslag [1] has shown that G is an R-group which cannot be embedded in a divisible R-group. Thus G can not be embedded in a divisible O-group, because every O-group is an R-group (Fuchs [2], p. 61). We show that G cannot be linearly ordered.

LEMMA 1. $ac^2a \neq ca^2c$ *in* G.

Proof. We use a normal form argument. (See Neumann [3] for the theory of normal form in a free product with amalgamation.)

Let S be a system of left coset representatives of A with respect to H such that both $b^{1/4}$ and $b^{1/2}$ belong to S. $T = \{1, c\}$ is a system of left coset representatives of C with respect to H. (Observe that $b^{1/4}$ and $b^{1/2}$ lie in different cosets of H because every non-identity element of H has a non-trivial power of a in its unique representation in A.)

Now

$$ac^2a = ac^2a^2a^{-1} = aba^{-1} = b^{1/2} .$$

Since $b^{1/2} \in S$, $b^{1/2}$ is the normal form of ac^2a in G (with respect to S, T and H).

But

$$\begin{aligned} ca^2c &= ca^2ba^{-2}a^2b^{-1}c \\ &= cb^{1/4}a^2b^{-1}c \\ &= cb^{1/4}a^2(c^2a^2)^{-1}c \\ &= cb^{1/4}c^{-1} \\ &= cb^{1/4}cc^{-2} . \end{aligned}$$

Since $b^{1/4} \in S$, $c \in T$ and $c^{-2} \in H$, $cb^{1/4}cc^{-2}$ is the normal form of ca^2c in G .

The next lemma shows that G is not an O-group.

LEMMA 2. *Let K be an O-group with elements x and y which satisfy*

(1) $$x^{-1}y^2x = y^2x^2y^2 .$$

Then $xy^2x = yx^2y$.

Proof. We show that neither

(2) $$xy^2x < yx^2y$$

nor

(3) $$xy^2x > yx^2y$$

hold in K .

If we assume that (2) holds, we have

$$\begin{aligned} xy^2x < yx^2y &\Rightarrow xy^2x < y^{-1}x^{-1}y^2xy^{-1} \quad \text{by (1)} \\ &\Rightarrow xy^2x < y^{-1}x^{-2}xy^2xy^{-1} \\ &\Rightarrow xy^2x < y^{-1}x^{-2}yx^2yy^{-1} \quad \text{by (2)} \\ &\Rightarrow xy^2 < y^{-1}x^{-2}yx \\ &\Rightarrow xy^2 < y^{-1}x^{-2}y^{-1}y^2x \\ &\Rightarrow xy^2 < x^{-1}y^{-2}x^{-1}y^2x \quad \text{by (2)} \\ &\Rightarrow xy^2 < x^{-1}y^{-2}y^2x^2y^2 \quad \text{by (1)} \\ &\Rightarrow xy^2 < xy^2 \quad \text{- impossible.} \end{aligned}$$

So $xy^2x \nless yx^2y$.

Now assume that (3) holds. By substituting $>$ for $<$ and (3) for (2) in the above argument, the validity of this argument is not affected. So $xy^2x \ngtr yx^2y$. Hence $xy^2x = yx^2y$ and Lemma 2 is proven.

Finally, we observe that a and c in G satisfy (1). (Because

$b = c^2a^2$ and $a^{-1}ba = b^2$ imply $a^{-1}(c^2a^2)a = (c^2a^2)^2$; that is $a^{-1}c^2a = c^2a^2c^2$.) So, if G were an O-group, then, by Lemma 2, $ac^2a = ca^2c$ would hold, contrary to Lemma 1, so G is not an O-group.

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